CS525 FINAL - FALL 2010 - 12/23/10.

1. (a)

	x_1	x_2	x_3	1
x_4 =	1	0	1	-2
x_5 =	4	-1	0	1
z:	-2	3	1	0

$x_5 = 0$.

↓

	x_1	x_5	x_3	1
x_4 =	1	0	1	-2
x_2 =	4	-1	0	1
z:	10	-3	1	0

Scheme II

	x_1	x_3	1
x_4 =	1	1	-2
x_2 =	4	0	1
z:	10	1	0

dual simplex

	x_1	x_4	1
x_3 =	-1	1	2
x_2 =	4	0	1
z:	9	1	2

optimal!

$$x = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \qquad z^* = 2.$$

②

1(b) $\min_{(x,\epsilon)} \epsilon \quad \text{s.t.} \quad -\epsilon e \le Ax - b \le \epsilon e$

$$\Longleftrightarrow \quad \min_{(x,\epsilon)} \epsilon \quad \text{s.t.} \quad \begin{aligned} Ax + \epsilon e &\ge b \\ -Ax + \epsilon e &\ge -b. \end{aligned}$$

	x_1	x_2	ϵ	1
$x_3 =$	1	3	1	-2
$x_4 =$	2	-1	1	-2
$x_5 =$	4	7	1	-10
$x_6 =$	-1	-3	1	2
$x_7 =$	-2	1	1	2
$x_8 =$	-4	-7	1	10
$z =$	0	0	1	0

Since x_1, x_2 are free (so is ϵ actually, but we can treat it as a nonnegative variable) we would start by doing "Scheme II" pivots to move x_1, x_2 to become row labels

2.

	x_1	x_2	1	t
$x_3 =$	-1	0	-4	-1
$x_4 =$	1	1	0	2
$z =$	1	2	0	0

eg any $t \leq -4$ clearly gives a nonempty feasible region

Can get "initial t" by eyeballing, or solving a "Phase I problem" of the form

$$\min_{(t, x, x_0)} x_0 \quad \text{s.t.} \quad Ax + x_0 e \geq b + th$$
$$x \geq 0, \; x_0 \geq 0, \; t \text{ free.}$$

Following the second strategy, we have the following:

	x_1	x_2	t	x_0	1
$x_3 =$	-1	0	-1	1	-4
$x_4 =$	1	1	2	1	0
$z =$	1	2	0	0	0
$z_0 =$	0	0	0	1	0

t free – exchange to side, as per Scheme II

↓

	x_1	x_2	x_3	x_0	1
$t =$	-1	0	-1	1	-4
$x_4 =$	-1	1	-2	3	-8
$z =$	1	2	0	0	0
$z_0 =$	0	0	0	1	0

↓ move to bottom

	x_1	x_2	x_3	x_0	1
$x_4 =$	-1	1	-2	3	-8
$z =$	1	2	0	0	0
$z_0 =$	0	0	0	1	0
$t =$	-1	0	-1	1	-4

dual simplex on z_0 row

↓

	x_1	x_4	x_3	x_0	1
$x_2 =$	1	1	2	-3	8
$z =$	3	2	4	-6	16
$z_0 =$	0	0	0	0	0
$t =$	-1	0	-1	1	-4

optimal with $t = -4$

Return to original tableau, and solve for $t = -4$

/over

	x_1	x_2	1	t
$x_3 =$	-1	0	-4	-1
$x_4 =$	1	1	0	2
$z:$	1	2	0	0

$t = -4$: apply dual simplex

	x_4	x_2	1	t
$x_3 =$	-1	1	-4	1
$x_1 =$	1	-1	0	-2
$z:$	1	1	0	-2

not optimal yet.

	x_4	x_3		
$x_2 =$	1	1	4	-1
$x_1 =$	0	-1	-4	-1
$z:$	2	1	4	-3

optimal for $4 - t \geq 0 \Rightarrow t \leq 4$

$-4 - t \geq 0 \Rightarrow t \leq -4$

$\Rightarrow$ for $t \in (-\infty, -4]$,

solution $x(t) = \begin{pmatrix} -4-t \\ 4-t \end{pmatrix}$

$z(t) = 4 - 3t$

as t increases through -4, second row of tableau becomes infeasible. According to dual simplex there is no suitable pivot column $\Rightarrow$ the problem is infeasible for $t \in (-4, \infty)$.

Conclusion

t	$x(t)$	$z(t)$
$(-\infty, -4]$	$\begin{pmatrix} -4-t \\ 4-t \end{pmatrix}$	$4 - 3t$
$(-4, +\infty)$	infeasible.	

3. (a) $2x_1 - 2x_2 - 2u_1 + 2 = 0$

$-2x_1 + 8x_2 - u_1 = 0$

$0 \le 2x_1 + x_2 - 4 \perp u_1 \ge 0$

(b) Set $\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ u_1 \end{pmatrix}$

$$\begin{array}{r|ccc|c|} & z_1 & z_2 & z_3 & 1 \\ \hline w_1 = & 2 & -2 & -2 & 2 \\ w_2 = & -2 & 8 & -1 & 0 \\ w_3 = & 2 & 1 & 0 & -4 \\ \hline \end{array}$$

pivot z_1, z_2 to side, pivot w_1, w_2 to top and delete.

$$\begin{array}{r|ccc|c|} & w_1 & z_2 & z_3 & 1 \\ \hline z_1 = & \frac{1}{2} & 1 & 1 & -1 \\ w_2 = & -1 & 6 & -3 & 2 \\ w_3 = & 1 & 3 & 2 & -6 \\ \hline \end{array}$$

$$\begin{array}{r|cc|c|} & w_2 & z_3 & 1 \\ \hline z_1 = & \frac{1}{6} & \frac{3}{2} & -\frac{4}{3} \\ z_2 = & \frac{1}{6} & \frac{1}{2} & -\frac{1}{3} \\ w_3 = & \frac{1}{2} & \frac{7}{2} & -7 \\ \hline \end{array}$$

rearrange rows, add artificial

$$\begin{array}{r|cc|c|l} & z_3 & z_0 & 1 & \\ \hline w_3 = & \frac{7}{2} & 1 & -7 & \leftarrow \\ z_1 = & \frac{3}{2} & 1 & -\frac{4}{3} & \\ z_2 = & \frac{1}{2} & 1 & -\frac{1}{3} & \\ \hline \end{array}$$

Phase 1 pivot

$$\begin{array}{c|cc|c}
 & z_3 & w_3 & 1 \\
\hline
z_0 = & -\frac{7}{2} & 1 & 7 \\
z_2 = & -2 & 1 & \frac{17}{3} \\
z_1 = & -3 & 1 & \frac{20}{3}
\end{array}$$

$$\begin{array}{c|cc|c}
 & z_0 & w_3 & 1 \\
\hline
z_3 = & -\frac{2}{7} & \frac{2}{7} & 2 \\
z_1 = & \frac{4}{7} & \vdots & \frac{5}{3} \\
z_2 = & \frac{6}{7} & \vdots & \frac{2}{3}
\end{array}$$

$$\frac{17}{3} - \frac{-14}{-7/2} = \frac{17}{3} - 4 = \frac{5}{3}$$

$$\frac{20}{3} - \frac{-21}{-7/2} = \frac{20}{3} - 6 = \frac{2}{3}$$

Solution $x = \begin{pmatrix} 5/3 \\ 2/3 \end{pmatrix}$ $u = (2)$

(c) If the constraint becomes an equality, KKT conditions are:

$$2x_1 - 2x_2 - 2u_1 + 2 = 0$$
$$-2x_1 + 8x_2 - u_1 = 0$$
$$2x_1 + x_2 - 4 = 0$$

which is still satisfied by $x = \begin{pmatrix} 5/3 \\ 2/3 \end{pmatrix}$ and $u = (2)$

so the solution remains the same.

4. (a) Expected loss for Player I:

$$x'Ay = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)$$

(b) If $y \neq (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ then we have that at least one of $y_2 - y_3$, $y_3 - y_1$, $y_1 - y_2$ is strictly negative. Set the component x_i of x corresponding to the most negative such value to 1, and the other two components of x to zero, to get a negative expected loss for Player I.

(c) $x' A\bar{y} = x' 0 = \bar{x}' A\bar{y}$ for any strategy vector x

$\bar{x}' A y = 0' y = 0$. for any strategy vector y

so the conditions for a Nash eq hold, ie

$x' A\bar{y} \geq \bar{x}' A\bar{y}$ for all strategies x

$\bar{x}'(-A)y \geq \bar{x}'(-A)\bar{y}$ for all strategies y.

(d). NO. Since $\hat{y} \neq (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ we have as in (a) that there exists $\hat{x}$ s.t.

$$\hat{x}' A\hat{y} < 0.$$

Since $0 = \bar{x}' A\hat{y}$, Player I can improve their payoff by switching from $\bar{x}$ to $\hat{x}$, hence $(\bar{x}, \hat{y})$ cannot be a Nash equilibrium