(1) (a) write as:

$$\min \; -2x_1 + 3x_2$$
$$\text{st} \quad x_1 + x_2 \geq -2$$
$$-x_1 + 2x_2 = 1$$
$$-x_1 \geq -5$$
$$x_1 \geq 0$$

	x_1	x_2	1
$x_3 =$	1	1	2
→ $x_4 =$	-1	2	-1
$x_5 =$	-1	0	5
$z =$	-2	3	0

scheme II

	x_1	x_4	1
$x_3 =$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{5}{2}$
$x_2 =$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$x_5 =$	-1	0	5
$z =$	$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$

	x_1	1
$x_3 =$	$3/2$	$5/2$
$x_2 =$	$1/2$	$1/2$
$x_5 =$	-1	5
$z =$	$-1/2$	$3/2$

	x_1	1
$x_3 =$	$\frac{3}{2}$	$\frac{5}{2}$
→ $x_5 =$	-1	5
$z =$	$-\frac{1}{2}$	$\frac{3}{2}$
$x_2 =$	$\frac{1}{2}$	$\frac{1}{2}$

→

	x_5	1
$x_3 =$	$-\frac{3}{2}$	10
$x_1 =$	-1	5
$z =$	$\frac{1}{2}$	-1
$x_2 =$	$-\frac{1}{2}$	3

optimal:

$$x = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

$$z^* = -1$$

(b) $0 \le x_1 \perp u_1 - u_2 - u_3 + 2 \le 0$

$u_1 + 2u_2 - 3 = 0$

$0 \le u_1 \perp x_1 + x_2 + 2 \ge 0$

$-x_1 + 2x_2 - 1 = 0$

$0 \le u_3 \perp -x_1 + 5 \ge 0$

(c) $\max \; -2u_1 + u_2 - 5u_3$

s.t. $u_1 - u_2 - u_3 + 2 \le 0$

$u_1 + 2u_2 - 3 = 0$

$u_1 \ge 0, \; u_3 \ge 0.$

(d) use KKT conditions and known primal solution to obtain dual solution:

$x_1 > 0 \Rightarrow u_1 - u_2 - u_3 = -2$

$u_1 + 2u_2 = 3$

$x_1 + x_2 + 2 > 0 \Rightarrow u_1 = 0$

$\Rightarrow -u_2 - u_3 = -2$

$2u_2 = 3 \Rightarrow u_2 = \frac{3}{2}$

$\Rightarrow u_3 = -u_2 + 2 = \frac{1}{2}$

$\Rightarrow$ dual sol: $u^* = \begin{pmatrix} 0 \\ \frac{3}{2} \\ \frac{1}{2} \end{pmatrix}$

dual obj $= \frac{3}{2} - \frac{5}{2} = -1 =$ primal obj

2) P: min $0 \cdot x$ s.t. $Ax = b$, $-x \geq -e$, $x \geq 0$

D: max $b^T u - e^T v$ s.t. $A^T u - v \leq 0$, $v \geq 0$.

I is true

$\Rightarrow$ P has a solution

$\Rightarrow$ D has a solution with same optimal objective (strong duality)

$\Rightarrow$ $b^T u - e^T v \leq 0$ whenever $A^T u - v \leq 0$, $v \geq 0$

$\Rightarrow$ II is false

I is false

$\Rightarrow$ P is infeasible

$\Rightarrow$ D is either infeasible or unbounded (strong duality)

But since D is feasible ($u=0$, $v=0$ is a feasible point)

$\Rightarrow$ D is unbounded

$\Rightarrow$ $\exists (u,v)$ s.t. $b^T u - e^T v = 1$, $A^T u - v \leq 0$, $v \geq 0$

$\Rightarrow$ II true

(3) (a) $\min_{x_1,x_2,x_3} \; x_3$

st $\quad 2x_1 + 3x_2 \geq -2 + 3t$

$x_3 \geq x_1 + t$

$x_3 \geq -(x_1 + t)$

$(x_1, x_2, x_3) \geq 0$

standard form

$\min \; x_3$

st $\quad 2x_1 + 3x_2 \geq -2 + 3t$

$-x_1 + x_3 \geq t$

$x_1 + x_3 \geq -t$

$x \geq 0$

(b)

	x_1	x_2	x_3	1	t
$x_4 =$	2	3	0	2	-3
$x_5 =$	-1	0	1	0	-1
$x_6 =$	1	0	1	0	1
$z =$	0	0	1	0	0

already optimal for $t = 0$! solution $x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

objective $z = 0$

As t becomes slightly < 0, 3rd element of R.H.S. becomes negative. Dual simplex rules tell us to pivot on the (3,1) element

	x_6	x_2	x_3	1	t
$x_4 =$	2	3	-2	2	-5
$x_5 =$	-1	0	2	0	0
$x_1 =$	1	0	-1	0	-1
$z =$	0	0	1	0	0

optimal if $\quad 2 - 5t \geq 0 \Rightarrow t \leq \frac{2}{5}$

$\quad -t \geq 0 \Rightarrow t \leq 0$

$\Rightarrow$ Solution for $t \in (-\infty, 0]$ is $\quad x = \begin{pmatrix} -t \\ 0 \\ 0 \end{pmatrix}$

with objective $z = 0$

return to initial tableau and consider what happens as t becomes slightly > 0. Here the second element on the RHS becomes negative. Dual simplex rules tell us to pivot on the (2,3) element.

	x_1	x_2	x_5	1	t
$x_4 =$	2	3	0	2	-3
$x_3 =$	1	0	1	0	1
$x_6 =$	2	0	1	0	2
$z =$	1	0	1	0	1

optimal when $\quad 2 - 3t \geq 0 \Rightarrow t \leq \frac{2}{3}$

$\quad t \geq 0$

$\quad 2t \geq 0$

$\Rightarrow t \in [0, \frac{2}{3}]$

with solution $x = \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix}$ objective t.

As t increase past $\frac{2}{3}$, first element of RHS becomes negative. Dual simplex tells us to pivot on the (1,2) element

$$\begin{array}{c|ccc|cc} & x_1 & x_4 & x_5 & 1 & t \\ \hline x_2 = & -\frac{2}{3} & \frac{1}{3} & 0 & -\frac{2}{3} & 1 \\ x_3 = & 1 & 0 & 1 & 0 & 1 \\ x_6 = & 2 & 0 & 1 & 0 & 2 \\ \hline z = & 1 & 0 & 1 & 0 & 1 \end{array}$$

optimal for $\left.\begin{array}{l} -\frac{2}{3} + t \geq 0 \Rightarrow t \geq \frac{2}{3} \\ t \geq 0 \\ 2t \geq 0 \end{array}\right] \Rightarrow t \in \left[\frac{2}{3}, \infty\right)$

with solution $x = \begin{pmatrix} 0 \\ -\frac{2}{3} + t \\ t \end{pmatrix}$ $\quad z(t) = t$

Summary

t	$x(t)$	$z(t)$
$(-\infty, 0]$	$\begin{pmatrix} -t \\ 0 \\ 0 \end{pmatrix}$	0
$\left[0, \frac{2}{3}\right]$	$\begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix}$	t
$\left[\frac{2}{3}, \infty\right)$	$\begin{pmatrix} 0 \\ -\frac{2}{3} + t \\ t \end{pmatrix}$	t

4 (a) $\min\limits_{x_1, x_2} \; x_1^2 - 2x_1 + x_2$

st $\quad x_2 + x_1 \geq 5$

$(x_1, x_2) \geq 0$

QP in standard form with $Q = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad c = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$A = [1 \;\; 1] \quad b = [5]$.

(b) KKT:
$$0 \leq 2x_1 - 2 - u_1 \perp x_1 \geq 0$$
$$0 \leq 1 - u_1 \perp x_2 \geq 0$$
$$0 \leq x_1 + x_2 - 5 \perp u_1 \geq 0$$

(c) LCP form $\quad z = \begin{pmatrix} x_1 \\ x_2 \\ u_1 \end{pmatrix} \quad M = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \quad q = \begin{bmatrix} -2 \\ 1 \\ -5 \end{bmatrix}$

	z_1	z_2	z_3	z_0	1
$w_1 =$	2	0	-1	1	-2
$w_2 =$	0	0	-1	1	1
$w_3 =$	1	1	0	1	-5

Phase I

	z_1	z_2	z_3	w_3	1
$w_1 =$	1	-1	-1	1	3
$w_2 =$	-1	-1	-1	1	6
$z_0 =$	-1	-1	0	1	5

	z_1	z_2	w_1	w_3	1
$z_3 =$	1	-1	-1	1	3
$w_2 =$	-2	0	1	0	3
$z_0 =$	-1	-1	0	1	5

$\rightarrow$

	w_2	z_2	w_1	w_3	1
$z_3 =$	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	1	$\frac{9}{2}$
$z_1 =$	$-\frac{1}{2}$	0	1	0	$\frac{3}{2}$
$z_0 =$	$\frac{1}{2}$	-1	$-\frac{1}{2}$	1	$\frac{7}{2}$

	w_2	z_0	w_1	w_3	1
$z_3 =$		1			1
$z_1 =$		0			$\frac{3}{2}$
$z_2 =$	$\frac{1}{2}$	-1	$-\frac{1}{2}$	1	$\frac{7}{2}$

Solution is $z = \begin{pmatrix} 3/2 \\ 0 \\ 9/2 \end{pmatrix} \Rightarrow$ $x_1 = z_1 = \frac{3}{2}$

$x_2 = z_2 = \frac{7}{2}$

$u_1 = z_3 = 1$

Satisfies K.K.T

(d) The QP is convex, so the KKT point is a global solution