525 FINAL EXAM FALL 2014

SOLUTIONS

1. (a)

	x_1	x_2	1
$x_3 =$	2	3	2
$x_4 =$	1	1	-6
$x_5 =$	-1	0	10
$z =$	1	2	0

Scheme II $\longrightarrow$

	x_1	x_4	1
$x_3 =$	-1	3	20
$x_2 =$	-1	1	6
$x_5 =$	-1	0	10
$z =$	-1	2	12

$\downarrow$ remove column $x_4 = 0$

	x_1	1
$x_3 =$	-1	20
$x_2 =$	-1	6
$x_5 =$	-1	10
$z =$	-1	12

move free var x_2 to bottom $\longleftarrow$

	x_1	1
$x_3 =$	-1	20
$x_5 =$	-1	10
$z =$	-1	12
$x_2 =$	-1	6

$\downarrow$ simplex (Phase II)

	x_5	1
$x_3 =$	1	10
$x_1 =$	-1	10
$z =$	1	2
$x_2 =$	1	-4

optimal! $x = \begin{pmatrix} 10 \\ -4 \end{pmatrix}$.

(b) Dual:

$$\max_{u_1,u_2,u_3} \; -2u_1 + 6u_2 - 10u_3$$

$$\text{s.t.} \quad 2u_1 + u_2 - u_3 \le 1$$

$$3u_1 + u_2 = 2$$

$$u_1 \ge 0, \; u_3 \ge 0 \quad (u_2 \text{ free})$$

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	x_1	x_2	x_3	1
$x_4 =$	1	0	3	-3
$x_5 =$	0	2	2	-5
$z =$	3	5	0	0
$z_0 =$	2	-1	0	0

not primal feasible, but it is dual feasible for $\theta = 0$, so do dual simplex pivot

$\downarrow$

	x_1	x_2	x_5	1
$x_4 =$	1	-3	$\frac{3}{2}$	$9/2$
$x_3 =$	0	-1	$\frac{1}{2}$	$5/2$
$z =$	3	5	0	0
$z_0 =$	2	-1	0	0

primal feasible! optimal if

$3 + 2\theta \geq 0 \Rightarrow \theta \geq -3/2$

$5 - \theta \geq 0 \Rightarrow \theta \leq 5$

So for $\theta \in [-3/2, 5]$ have solution $x = \begin{pmatrix} 0 \\ 0 \\ 5/2 \end{pmatrix}$

As θ drops below $-3/2$, we seek a pivot in the first column but we can find no suitable pivot, - all entries are nonnegative. Hence problem is unbounded for $\theta < -3/2$.

As θ increases through 5, use second column as pivot column. Ratio test shows that $(1,2)$ is the pivot location.

After pivoting, obtain

	x_1	x_4	x_5	1
$x_2 =$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{3}{2}$
$x_3 =$	$-\frac{1}{3}$	$\frac{1}{3}$	0	1
$z =$	$\frac{14}{3}$	$-\frac{5}{3}$	$\frac{5}{2}$	$\frac{15}{2}$
$z_0 =$	$\frac{5}{3}$	$\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{3}{2}$

which is optimal if

$$\left.\begin{aligned} \tfrac{14}{3} + \tfrac{5}{3}\theta \geq 0 &\implies \theta \geq -\tfrac{14}{5} \\ -\tfrac{5}{3} + \tfrac{1}{3}\theta \geq 0 &\implies \theta \geq 5 \\ \tfrac{5}{2} - \tfrac{1}{2}\theta \geq 0 &\implies \theta \leq 5 \end{aligned}\right\} \text{ all true only if } \theta = 5$$

Hence this tableau reveals another solution for $\theta = 5$, namely $x = \begin{pmatrix} 0 \\ 3/2 \\ 1 \end{pmatrix}$

As θ increases above 5, the 'reduced cost' in the third column becomes negative, but there is no suitable pivot row, so problem is again unbounded for $\theta > 5$

Summarizing:

θ	x^*	z
$(-\infty, -\frac{3}{2})$	—	(unbounded)
$[-\frac{3}{2}, 5]$	$\begin{pmatrix} 0 \\ 0 \\ 5/2 \end{pmatrix}$	0
5	$\begin{pmatrix} 0 \\ 3/2 \\ 1 \end{pmatrix}$	0
$(5, \infty)$	—	(unbounded)

3. (P) : $\min 0^T x$ s.t. $Ax \geq e$, $x \geq 0$

(D) $\max e^T u$ s.t. $A^T u \leq 0$, $u \geq 0$

where $e = (1, 1, \ldots, 1)^T$.

I true $\Rightarrow$ (P) has a feasible point, and thus a solution, with optimal objective zero

$\Rightarrow$ (D) also has a solution, and all feasible points u for (D) must have

$$A^T u \leq 0, \quad u \geq 0, \quad e^T u \leq 0.$$

The last two conditions combined show that $u = 0$ is the only feasible point for (D)

$\Rightarrow$ II is not true

I is false $\Rightarrow$ (P) is infeasible

$\Rightarrow$ (D) is either infeasible or unbounded but it cannot be infeasible, since $u = 0$ is clearly a feasible point

$\Rightarrow$ (D) is unbounded

$\Rightarrow$ There exists u such that

$$A^T u \leq 0, \quad u \geq 0, \quad e^T u > 0$$

$\Rightarrow$ II is true

Hence we have shown that exactly one of the statements I and II are true!

4. (a) since z and $Mz+q$ are both nonnegative vectors, their inner product $z^T(Mz+q)$, which is the QP objective, must also be nonnegative

(b) $Q = M+M^T$, so QP is

$$\min_z \ \tfrac{1}{2} z^T (M+M^T) z + q^T z \quad \text{st} \quad Mz+q \geq 0, \ z \geq 0$$

(c) if z^* solves the LCP we have

$$(z^*)^T (Mz^* + q) = 0.$$

In addition, z^* is feasible for the QP since $z^* \geq 0$, $Mz^* + q \geq 0$

Since we showed in (a) that all other feasible points have nonnegative objectives, z^* must be a solution

(d) $0 \leq (M+M^T) z + q - M^T u \perp z \geq 0$

$0 \leq Mz+q \perp u \geq 0$

(e) Setting $z = z^*$, $u = z^*$, the KKT conditions become

$$0 \leq Mz^* + q \perp z^* \geq 0$$

$$0 \leq Mz^* + q \perp z^* \geq 0$$

which are both satisfied, since z^* solves the LCP