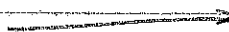


SOLUTIONS

1. (a)

	x_1	x_2	1
x_3	2	3	2
x_4	1	1	-6
x_5	-1	0	10
z	1	2	0

Scheme II

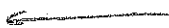


	x_1	x_4	1
x_3	-1	3	20
x_2	-1	1	6
x_5	-1	0	10
z	-1	2	12

↓ remove column $x_4=0$

	x_1	1
x_3	-1	20
x_5	-1	10
z	-1	12
x_2	-1	6

move free var x_2 to bottom



	x_1	1
x_3	-1	20
x_2	-1	6
x_5	-1	10
z	-1	12

↓ Simplex (Phase II)

	x_5	1
x_3	1	10
x_1	-1	10
z	1	2
x_2	1	-4

Optimal! $x = \begin{pmatrix} 10 \\ -4 \end{pmatrix}$

↳ Dual: $\max_{u_1, u_2, u_3} -2u_1 + 6u_2 - 10u_3$

s.t. $2u_1 + u_2 - u_3 \leq 1$

$3u_1 + u_2 = 2$

$u_1 \geq 0, u_3 \geq 0$ (u_2 free)

2

	x_1	x_2	x_3	1
$x_4 =$	1	0	3	-3
$x_5 =$	0	2	2	-5
$z =$	3	5	0	0
$z_0 =$	2	-1	0	0

not primal feasible,
 but it is dual feasible
 for $\theta = 0$, so do dual
simplex pivot



	x_1	x_2	x_5	1
$x_4 =$	1	-3	$\frac{3}{2}$	$\frac{9}{2}$
$x_3 =$	0	-1	$\frac{1}{2}$	$\frac{5}{2}$
$z =$	3	5	0	0
$z_0 =$	2	-1	0	0

primal feasible!
 optimal if

$$3 + 2\theta \geq 0 \Rightarrow \theta \geq -\frac{3}{2}$$

$$5 - \theta \geq 0 \Rightarrow \theta \leq 5$$

So for $\theta \in [-\frac{3}{2}, 5]$ have

solution $x = \begin{pmatrix} 0 \\ 0 \\ 5/2 \end{pmatrix}$

As θ drops below $-\frac{3}{2}$, we seek a pivot in the first column but we can find no suitable pivot, - all entries are non-negative. Hence problem is unbounded for $\theta < -\frac{3}{2}$.

As θ increases through 5, use second column as pivot column. Ratio test shows that (1,2) is the pivot location.

After pivoting, obtain

	x_1	x_4	x_5	
$x_2 =$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{15}{2}$
$x_3 =$	$-\frac{1}{3}$	$\frac{1}{3}$	0	1
$z =$	$\frac{14}{3}$	$-\frac{4}{3}$	$\frac{5}{2}$	$\frac{15}{2}$
$z_0 =$	$\frac{5}{3}$	$\frac{1}{3}$	$-\frac{1}{2}$	$\frac{13}{3}$

which is optimal if

$$\left. \begin{aligned} \frac{14}{3} + \frac{5}{3}\theta \geq 0 &\Rightarrow \theta \geq -\frac{14}{5} \\ -\frac{4}{3} + \frac{1}{3}\theta \geq 0 &\Rightarrow \theta \geq 4 \\ \frac{5}{2} - \frac{1}{2}\theta \geq 0 &\Rightarrow \theta \leq 5 \end{aligned} \right\} \begin{array}{l} \text{all true} \\ \text{only if} \\ \theta = 5 \end{array}$$

Hence the tableau reveals another solution for $\theta = 5$, namely $x = \begin{pmatrix} 0 \\ 3/2 \\ 1 \end{pmatrix}$

As θ increases above 5, the reduced cost in the third column becomes negative, but there is no suitable pivot row, so problem is again unbounded for $\theta > 5$

Summarizing:

θ	x	z
$(-\infty, -\frac{3}{2})$	—	(unbounded)
$[-\frac{3}{2}, 5]$	$\begin{pmatrix} 0 \\ 0 \\ 5/2 \end{pmatrix}$	0
5	$\begin{pmatrix} 0 \\ 3/2 \\ 1 \end{pmatrix}$	0
$(5, \infty)$	—	(unbounded)

3. (P) : $\min c^T x$ s.t. $Ax \leq b, x \geq 0$

(D) $\max e^T u$ s.t. $A^T u \leq c, u \geq 0$

where $e = (1, 1, \dots, 1)^T$.

I true \Rightarrow (P) has a feasible point, and thus a solution, with optimal objective $z = 0$

\Rightarrow (D) also has a solution, and all feasible points u for (D) must have

$A^T u \leq c, u \geq 0, e^T u \leq 0$.

The last two conditions combined show that $u = 0$ is the only feasible point for (D)

\Rightarrow II is not true

I is false \Rightarrow (P) is infeasible

\Rightarrow (D) is either infeasible or unbounded but it cannot be infeasible, since $u = 0$ is clearly a feasible point

\Rightarrow (D) is unbounded

\Rightarrow There exists u such that

$A^T u \leq c, u \geq 0, e^T u > 0$

\Rightarrow II is true

Hence we have shown that exactly one of the statements I and II are true!

4. (a) since z and $Mz+g$ are both nonnegative vectors, their inner product $z^T(Mz+g)$, which is the QP objective, must also be nonnegative

(b) $Q = M + M^T$, so QP is

$$\min_z \frac{1}{2} z^T (M + M^T) z + g^T z \quad \text{st} \quad Mz + g \geq 0, z \geq 0$$

(c) if z^* solves the LCP we have

$$(z^*)^T (Mz^* + g) = 0.$$

In addition, z^* is feasible for the QP since $z^* \geq 0, Mz^* + g \geq 0$

Since we showed in (a) that all other feasible points have nonnegative objectives, z^* must be a solution

$$\begin{aligned} \text{(d)} \quad 0 &\leq (M + M^T) z + g - M^T u \perp z \geq 0 \\ 0 &\leq Mz + g \perp u \geq 0 \end{aligned}$$

(e) Setting $z = z^*, u = z^*$, the KKT conditions become

$$0 \leq Mz^* + g \perp z^* \geq 0$$

$$0 \leq Mz^* + g \perp z^* \geq 0$$

which are both satisfied, since z^* solves the LCP