Final Examination

CS 525 - Fall 2014

Wednesday, Dec 17, 2014, 12:25p-2:25p.

No electronic devices, notes, or books allowed, except that you may bring one standard-size sheet of paper, handwritten on both sides, into the test. **Give reasoning and justify all your answers**, quoting any theorems you use.

1. (a) Solve the following linear program. If it is unbounded, give a direction of unboundedness. (Hint: Use Scheme II.)

$$\begin{aligned} \min_{x_1,x_2} \ & x_1 + 2x_2 \\ \text{subject to} \quad & 2x_1 + 3x_2 \geq -2, \\ & x_1 + x_2 = 6, \\ & 0 \leq x_1 \leq 10, \quad (x_2 \text{ free}). \end{aligned}$$

   (b) Write down the dual of the problem in (a).

2. Consider the following LP with 3 variables and two constraints, with a parametric objective:

$$\begin{aligned} \min_{x_1,x_2,x_3} \ & (3+2\theta)x_1 + (5-\theta)x_2 \\ \text{subject to} \quad & x_1 + 3x_3 \geq 3, \\ & 2x_2 + 2x_3 \geq 5, \\ & (x_1, x_2, x_3) \geq 0. \end{aligned}$$

   (Note that x_3 does not appear in the objective, x_2 does not appear in the first constraint, and x_1 does not appear in the second constraint.) Draw up a table showing solutions and optimal objective values of this problem for all values of θ.

3. Given a matrix A, consider the following two statements:

   I: There exists a vector x such that $Ax > 0$ and $x \geq 0$;

   II: There exists a vector u such that $A^T u \leq 0$ and $u \geq 0$ and $u \neq 0$.

   Show that exactly one of these two statements is true.

4. Consider the following LCP, some matrix M of dimension $n \times n$ that is positive semidefinite but not necessarily symmetric:

$$\text{LCP:} \quad 0 \leq z \perp Mz + q \geq 0,$$

   where q is a given vector with n elements. Consider too the following QP that is constructed from the LCP data objects M and q:

$$\text{QP:} \quad \min_z \; z^T(Mz + q) \quad \text{subject to } Mz + q \geq 0, \; z \geq 0.$$

   (a) Explain why the objective of the QP is nonnegative at all feasible points z.

   (b) Write the objective function for the QP above in the conventional form:
   $$\frac{1}{2} z^T Q z + q^T z,$$
   that is, define the *symmetric* matrix Q such that this objective is identical to $z^T(Mz + q)$.

   (c) Suppose that z^* is a solution of the LCP. Explain carefully why z^* must also be a solution of the QP, and write down the optimal value of the QP objective.

   (d) Using the form of the QP objective in (a), write down the KKT conditions for the QP.

   (e) Suppose as in (b) that the LCP has a solution z^*. Show that the KKT conditions for the QP can be satisfied by setting the Lagrange multiplier vector u^* corresponding to the constraint $Mz + q \geq 0$ to be equal to z^*.