525 Spring 2014 Q2 (alternative solution)

I : $Bx \geq 0$, $Bx \neq 0$ for some x

II : $B^T y = 0$, $y > 0$ for some y

Note that II is equivalent to

II′ : $B^T y = 0$, $y \geq e$, where $e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

(because given any y with all positive elements, we can apply a scaling factor to increase all those elements above 1).

Define

P : $\min 0^T y$ s.t. $B^T y = 0$, $y \geq e$

D : $\max e^T v$ s.t. $Bu + v = 0$, $v \geq 0$.

II′ true $\Rightarrow$ P feasible and has a solution with optimal objective 0

$\Rightarrow$ (weak duality) all feasible points (u, v) for D have $Bu + v = 0$, $v \geq 0$, $e^T v \leq 0$

$\Rightarrow$ last two condition shows that any feasible point (u, v) for D must have $\underline{\underline{v = 0}}$

$\Rightarrow$ for any u s.t. $Bu = -v \leq 0$, we must have $Bu = 0$.

$\Rightarrow$ By setting $x = -u$, we see that for any x with $Bx \geq 0$ we must have $Bx = 0$

$\Rightarrow$ I is false

II false $\Rightarrow$ P is infeasible

$\Rightarrow$ (strong duality) D is either infeasible or unbounded

$\Rightarrow$ since D is clearly feasible (by setting $u=0, v=0$) it must be unbounded

$\Rightarrow$ can find (u,v) with

$$e^T v > 0, \quad Bu = -v \quad , v \geq 0$$

$\Rightarrow$ By defining $x = -u$, we have

$$Bx = -Bu = v \geq 0,$$

$$e^T Bx = e^T v > 0$$

$\Rightarrow$ I is true