525 FINAL SPRING 2014

1

	x_1	x_2	1
$x_3 =$	$\frac{1}{2}$	1	$-\frac{1}{2}$
$x_4 =$	1	-3	3
$z =$	2	3	0
$z_0 =$	0	1	0

Try solving for $t=0$ first. Can use dual simplex to choose pivot $(1,2)$

	x_1	x_3	1
$x_2 =$	$-\frac{1}{2}$	1	$\frac{1}{2}$
$x_4 =$	$\frac{5}{2}$	-3	$\frac{3}{2}$
$z =$	$\frac{1}{2}$	3	$\frac{3}{2}$
$z_0 =$	$-\frac{1}{2}$	1	$\frac{1}{2}$

*

Optimal for $\frac{1}{2} - \frac{1}{2}t \geq 0$, $3 + t \geq 0$

ie $t \leq 1$, $t \geq -3$

On interval $t \in [-3, 1]$, solution is $x(t) = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$

with objective $z(t) = \frac{3}{2} + \frac{1}{2}t$

When t decreases through -3, the reduced cost in column 2 becomes negative. Pivot on $(2,2)$ element.

	x_1	x_4	
$x_2 =$	$\frac{1}{3}$	$-\frac{1}{3}$	1
$x_3 =$	$\frac{5}{6}$	$-\frac{1}{3}$	$\frac{1}{2}$
$z =$	3	-1	3
$z_0 =$	$\frac{1}{3}$	$-\frac{1}{3}$	1

optimal when

$3 + \frac{1}{3}t \geq 0 \Rightarrow t \geq -9$

$-1 - \frac{1}{3}t \geq 0 \Rightarrow t \leq -3$

On $t \in [-9,-3]$ solution is $x(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
with objective value $z(t) = 3+t$

As t decreases through -9, first reduced cost becomes negative but there is no suitable pivot - problem is unbounded for $t < -9$.

Returning to tableau $*$, as t increases through $+1$, reduced cost in column 1 becomes negative. Pivot on (1,1) element to obtain

$$\begin{array}{r|cc|c|}
 & x_2 & x_3 & 1 \\
\hline
x_1 = & -2 & 2 & 1 \\
x_4 = & -5 & 2 & 4 \\
\hline
z = & -1 & 4 & 2 \\
z_0 = & 1 & 0 & 0 \\
\hline
\end{array}$$

optimal when $-1+t \geq 0 \Rightarrow t \geq 1$
and $4 \geq 0 \Rightarrow$ all t.

So on interval $t \in [1, \infty)$, solution is $x(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
with objective 2

Summary:

t	$x(t)$	$z(t)$
$(-\infty, -9)$	unbounded	
$[-9,-3]$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$3+t$
$[-3,1]$	$\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$	$\frac{3}{2} + \frac{1}{2}t$
$[1,\infty)$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	2

(2) Consider the following LP and its dual

(P) $\min_{(y,\alpha)} -\alpha$ s.t. $-B^T y = 0,$

$y - \alpha e \geq 0$

(D) $\max_{(u,v)} 0^T u + 0^T v$ s.t. $-Bu + v = 0$

$e^T v = 1$

$v \geq 0.$

Can write D equivalently as

(D') $\max_{u} 0^T u$ s.t. $Bu \geq 0, \; e^T(Bu) = 1.$

II is true $\Rightarrow$ (P) is feasible but unbounded

$\Rightarrow$ (D') is infeasible (by strong duality)

$\Rightarrow$ I is false

II is false $\Rightarrow$ (P) is feasible with solution $(y,\alpha) = (0,0)$

$\Rightarrow$ (D') is feasible (with zero objective
by strong duality

$\Rightarrow$ I is true

(2) (Alternative) This answer uses the facts that (a) any primal-dual solution pair satisfies KKT and (b) there exists a strictly complementary primal-dual solution pair (Thm 4.9.3)

Consider this primal-dual pair:

(P) $\min 0^T y$ s.t. $B^T y = 0,\ y \geq 0$

(D) $\max 0^T x$ s.t. $Bx \geq 0$.

Clearly both are feasible, with trivial solutions. The KKT conditions, satisfied by any primal-dual pair, are

$$0 \leq y \perp Bx \geq 0$$

$$B^T y = 0.$$

Theorem 4.9.3 says that there exist vectors $\hat{x}, \hat{y}$ satisfying KKT and in addition

$$\hat{y} + B\hat{x} > 0. \quad (1)$$

We reason as follows:

II is true $\Rightarrow$ $\exists$ a vector y with $B^T y = 0$, $y > 0$, which solves (P),

$\Rightarrow$ any vector x with $Bx \geq 0$ solves (D) and must have $Bx = 0$ by complementarity

$\Rightarrow$ I is false

II is false $\Rightarrow$ any y with $B^T y = 0, y \geq 0$ has $y_i = 0$ for at least one index i

$\Rightarrow$ given $(\hat{x}, \hat{y})$ that satisfy the KKT conditions and (1) we have $\hat{y}_i = 0$ and thus $(B\hat{x})_i > 0$ for this i

$\Rightarrow$ $\hat{x}$ satisfies $B\hat{x} \geq 0$ with $(B\hat{x})_i > 0$

$\Rightarrow$ I is true

3

(a)

$$
\begin{array}{r|ccc|c}
 & x_1 & x_2 & x_3 & 1 \\
\hline
x_4 = & 2 & 0 & 1 & -2 \\
x_5 = & 0 & 3 & \boxed{1} & -1 \\
\hline
z = & 4 & -5 & -1 & 0
\end{array}
$$

Scheme I: pivot x_3 to top and delete its column

$$
\begin{array}{r|ccc|c}
 & x_1 & x_2 & \cancel{x_5} & 1 \\
\hline
x_4 = & 2 & -3 & 1 & -1 \\
x_3 = & 0 & -3 & 1 & 1 \\
\hline
z = & 4 & -2 & -1 & -1
\end{array}
$$

$$
\begin{array}{r|cc|c}
 & x_1 & x_2 & 1 \\
\hline
x_4 = & 2 & -3 & -1 \\
x_3 = & 0 & -3 & 1 \\
\hline
z = & 4 & -2 & -1
\end{array}
$$

Scheme II: Pivot x_1 to side.

$$
\begin{array}{r|cc|c}
 & x_4 & x_2 & 1 \\
\hline
x_1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\
x_3 = & 0 & -3 & 1 \\
\hline
z = & 2 & 4 & 1
\end{array}
$$

Optimal! Solution is $x = \begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix}$

(3) (b)

$$\max_{u_1,u_2} \; 2u_1 + u_2$$
$$\text{s.t.} \quad 2u_1 = 4$$
$$3u_2 \le -5$$
$$u_1 + u_2 \le -1$$
$$u_1 \ge 0 \quad (u_2 \text{ free})$$

(c) KKT:

$$2u_1 = 4$$
$$0 \le -5 - 3u_2 \perp x_2 \ge 0$$
$$0 \le -1 - u_1 - u_2 \perp x_3 \ge 0$$
$$0 \le 2x_1 + x_3 - 2 \perp u_1 \ge 0$$
$$3x_2 + x_3 = 1$$

(d). Clearly from first KKT condition we have $u_1 = 2$.
Since $x_3 > 0$ in primal solution we must have

$$-1 - u_1 - u_2 = 0 \Rightarrow u_2 = -1 - u_1 = -3.$$

Checking other conditions:

$$0 \le -5 - 3u_2 = 4 \quad \checkmark$$
$$u_1 \ge 0 \quad \checkmark$$

So dual solution is $u = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

(e). Changing from -5 to -8 changes the second KKT condition to

$$0 \le -8 - 3u_2 \perp x_2 \ge 0$$

This and all other KKT conditions continue to be satisfied by the original primal and dual solutions, so primal solution does not change.

4 - (a) $A = \begin{bmatrix} 1 & 4 \\ 4 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 4 \\ 4 & 1 \end{bmatrix}$

(b) no, because $A + B \neq 0$.

(c) note that for $\bar{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\bar{y} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we have

$\bar{x}^T A \bar{y} = 1$, $\bar{x}^T B \bar{y} = 2$

to show Nash equilibrium, we have

$x^T A \bar{y} = (x_1\ x_2) \begin{pmatrix} 1 \\ 4 \end{pmatrix} = x_1 + x_2 + 3x_2 = 1 + 3x_2 \geq 1$ ✓

$\bar{x}^T B y = (2\ 4) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 2(y_1 + y_2) + 2y_2 = 2 + 2y_2 \geq 2$ ✓

(d) for $\bar{x} = \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix}$, $\bar{y} = \begin{pmatrix} 2/5 \\ 3/5 \end{pmatrix}$ we have

$\bar{x}^T A \bar{y} = (3/5\ \ 2/5) \begin{pmatrix} 1 & 4 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 2/5 \\ 3/5 \end{pmatrix} = (11/5\ \ 16/5) \begin{pmatrix} 2/5 \\ 3/5 \end{pmatrix} = \frac{70}{25} = \frac{14}{5}$

$\bar{x}^T B \bar{y} = (3/5\ \ 2/5) \begin{pmatrix} 2 & 4 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 2/5 \\ 3/5 \end{pmatrix} = (3/5\ \ 2/5) \begin{pmatrix} 16/5 \\ 11/5 \end{pmatrix} = \frac{70}{25} = \frac{14}{5}$

Checking Nash equilibrium properties.

$x^T A \bar{y} = (x_1\ x_2) \begin{pmatrix} 1 & 4 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 2/5 \\ 3/5 \end{pmatrix} = (x_1\ x_2) \begin{pmatrix} 14/5 \\ 14/5 \end{pmatrix} = \frac{14}{5} \geq \bar{x}^T A \bar{y}$ ✓

$\bar{x}^T B y = (3/5\ \ 2/5) \begin{pmatrix} 2 & 4 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (\frac{14}{5}\ \ \frac{14}{5}) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{14}{5} \geq \bar{x}^T B \bar{y}$ ✓

(e). $\bar{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\bar{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is also a Nash ~~eq~~ strategy.

(reasoning is the same as in (c))