Midterm Examination

CS 525 - Fall 2009

Monday, October 26, 2009, 7:15-9:15pm

Each question is worth the same number of points.

No electronic computing devices, notes, or books allowed, except that you may bring one standard-size sheet of paper, handwritten on both sides, into the test. **You need to give reasoning and justify all your answers**, citing the appropriate theorems where necessary.

1. (a) For the following matrix, find the linear dependence relations between its rows and between its columns, if any.

pivoting: 2
rows: 2
cols: 2

$$A = \begin{bmatrix} 1 & 4 & 3 & -2 \\ -1 & 2 & 1 & 0 \\ -4 & 2 & 0 & 2 \end{bmatrix}.$$

3 (b) What is the rank of the matrix in (a)?

(c) Using Jordan exchanges, find the inverse of this matrix:

pivots: 4
permutations: 2

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}.$$

5 (d) Give examples of a matrix A and right-hand sides b and c such that $Ax = b$ has no solutions while $Ax = c$ has infinitely many solutions.

2. Consider the following linear program:

$$\begin{array}{lrcrcrcr}
\min & x_1 & + & 2x_2 & + & 2x_3 & & \\
\text{subject to} & -2x_1 & + & x_2 & + & x_3 & \geq & 1, \\
& x_1 & - & x_2 & - & 2x_3 & \geq & -3, \\
& x_1 & , & x_2 & , & x_3 & \geq & 0.
\end{array}$$

6 (a) Write down the dual of this problem.

feasibility: 2 compl: 4 (b) Write down the KKT conditions for this problem.

8 (c) Using whatever techniques you wish, find solutions to both the primal and dual.

3. By using appropriate transformations and applying Scheme II, solve the following linear program, and write down the optimal value of the objective. (Hint: You should need no more than three pivots in total.)

convert to gen. form: 4
pivot eq constr: 4
pivot free: 4
simplex pivot: 4
correct x: 3
correct obj: 1

$$\begin{array}{lrcrcrcr}
\max & -x_1 & - & 4x_2 & + & x_3 & - & 2 \\
\text{subject to} & 2x_1 & + & 4x_2 & - & x_3 & \geq & 4, \\
& & & x_2 & + & x_3 & = & 8, \\
& 2x_1 & + & 6x_2 & & & \leq & 22, \\
& x_1 & & \text{unrestricted}, & & & & \\
& & & x_2 & , & x_3 & \geq & 0.
\end{array}$$

4. Consider the following linear program, where $c_1, c_2, \ldots, c_n$ are constants:

$$\max_{x_1, x_2, \ldots, x_n} \; c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_nx_n$$
$$\text{subject to } x_1 + x_2 + x_3 + \ldots + x_n = 1,$$
$$x_1, x_2, x_3, \ldots, x_n \geq 0.$$

dual: 3
soln: 3
unique: 2

(a) Write down the dual of this problem, and find the solution of the dual. Is it unique?

feas: 2 comp: 4 (b) Write down the KKT conditions for this problem.

4 (c) Use the KKT conditions to identify a solution $(x_1, x_2, \ldots, x_n)$ to the primal.

2 (d) Under what condition is the primal solution unique?

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1 (a)

		$v_1 =$ x_1	$v_2 =$ x_2	$v_3 =$ x_3	$v_4 =$ x_4
$-u_1$	$y_1 =$	1	4	3	-2
$-u_2$	$y_2 =$	-1	2	1	0
$-u_3$	$y_3 =$	-4	2	0	2

Pivot to find relationships between rows and columns

(1,1)

		$u_1 =$ y_1	$v_2 =$ x_2	$v_3 =$ x_3	$v_4 =$ x_4
$-v_1$	$x_1 =$	1	-4	-3	2
$-u_2$	$y_2 =$	-1	6	4	-2
$-u_3$	$y_3 =$	-4	18	12	-6

(2,4)

		$u_1 =$ y_1	$v_2 =$ x_2	$v_3 =$ x_3	$u_2 =$ y_2
$-v_1$	$x_1 =$	0	2	1	-1
$-v_4$	$x_4 =$	$-\frac{1}{2}$	3	2	$-\frac{1}{2}$
$-u_3$	$y_3 =$	-1	0	0	3

blocked!

rows: $A_{3\cdot} = -A_{1\cdot} + 3A_{2\cdot}$

columns: $A_{\cdot 2} = -2A_{\cdot 1} - 3A_{\cdot 4}$ and/or $A_{\cdot 3} = -A_{\cdot 1} - 2A_{\cdot 4}$

(b) rank is 2.

(c)

$$\begin{array}{c|ccc|} & x_1 & x_2 & x_3 \\ \hline y_1 = & 0 & 1 & -2 \\ y_2 = & 2 & 3 & -1 \\ y_3 = & 1 & -1 & 3 \\ \hline \end{array}$$

(1,2)

$$\begin{array}{c|ccc|} & x_1 & y_1 & x_3 \\ \hline x_2 = & 0 & 1 & 2 \\ y_2 = & 2 & 3 & 5 \\ y_3 = & 1 & -1 & 1 \\ \hline \end{array}$$

(3,1)

$$\begin{array}{c|ccc|} & y_3 & y_1 & x_3 \\ \hline x_2 = & 0 & 1 & 2 \\ y_2 = & 2 & 5 & 3 \\ x_1 = & 1 & 1 & -1 \\ \hline \end{array}$$

(2,3)

$$\begin{array}{c|ccc|} & y_3 & y_1 & y_2 \\ \hline x_2 = & -\frac{4}{3} & -\frac{7}{3} & \frac{2}{3} \\ x_3 = & -\frac{2}{3} & -\frac{5}{3} & \frac{1}{3} \\ x_1 = & \frac{5}{3} & \frac{8}{3} & -\frac{1}{3} \\ \hline \end{array}$$

permute rows:

$$\begin{array}{c|ccc|} & y_3 & y_1 & y_2 \\ \hline x_1 = & \frac{5}{3} & \frac{8}{3} & -\frac{1}{3} \\ x_2 = & -\frac{4}{3} & -\frac{7}{3} & \frac{2}{3} \\ x_3 = & -\frac{2}{3} & -\frac{5}{3} & \frac{1}{3} \\ \hline \end{array}$$

permute cols:

$$\begin{array}{c|ccc|} & y_1 & y_2 & y_3 \\ \hline x_1 = & \frac{8}{3} & -\frac{1}{3} & \frac{5}{3} \\ x_2 = & -\frac{7}{3} & \frac{2}{3} & -\frac{4}{3} \\ x_3 = & -\frac{5}{3} & \frac{1}{3} & -\frac{2}{3} \\ \hline \end{array}$$

$$A^{-1} = \begin{bmatrix} \frac{8}{3} & -\frac{1}{3} & \frac{5}{3} \\ -\frac{7}{3} & \frac{2}{3} & -\frac{4}{3} \\ -\frac{5}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

(d) $A = [0]$, $b = [1]$, $c = [0]$

2 (a) $\max \; u_1 - 3u_2$

$$\text{st.} \quad -2u_1 + u_2 \le 1$$
$$u_1 - u_2 \le 2$$
$$u_1 - 2u_2 \le 2$$
$$u_1 \ge 0, \; u_2 \ge 0$$

(b)

$$\left.\begin{array}{l} -2x_1 + x_2 + x_3 \ge 1 \\ x_1 - x_2 - 2x_3 \ge -3 \\ x_1 \ge 0, \; x_2 \ge 0, \; x_3 \ge 0 \end{array}\right\} \text{primal feasibility}$$

$$\left.\begin{array}{l} -2u_1 + u_2 \le 1 \\ u_1 - u_2 \le 2 \\ u_1 - 2u_2 \le 2 \\ u_1 \ge 0, \; u_2 \ge 0, \; u_3 \ge 0 \end{array}\right\} \text{dual feasibility}$$

$$\left.\begin{array}{l} [-2x_1 + x_2 + x_3 - 1]\,u_1 = 0 \\ [x_1 - x_2 - 2x_3 + 3]\,u_2 = 0 \\ [-2u_1 + u_2 - 1]\,x_1 = 0 \\ [u_1 - u_2 - 2]\,x_2 = 0 \\ [u_1 - 2u_2 - 2]\,x_3 = 0. \end{array}\right\} \text{complementarity}$$

(c) use dual simplex since all costs in primal are positive

		$u_3 =$	$u_4 =$	$u_5 =$	$w =$	
		x_1	x_2	x_3	1	
$-u_1$	$x_4 =$	-2	1	1	-1	←
$-u_2$	$x_5 =$	1	-1	-2	3	
1	$z =$	1	2	2	0	
			↑			

		$u_3 =$	$u_1 =$	$u_5 =$	$w =$
		x_1	x_4	x_3	1
$-u_4$	$x_2 =$	2	1	-1	1
$-u_2$	$x_5 =$	-1	-1	-1	2
1	$z =$	5	2	0	2

optimal! primal sol: $x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ dual sol: $u = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

3. $- \min \quad x_1 + 4x_2 - x_3 + 2$

$$\text{st.} \quad 2x_1 + 4x_2 - x_3 \geq 4$$
$$x_2 + x_3 = 8$$
$$-2x_1 - 6x_2 \geq -22$$
$$x_1 \text{ free}, \quad x_2 \geq 0, \quad x_3 \geq 0$$

		x_1 (free)	x_2	x_3	1
	$x_4 =$	2	4	-1	-4
eq. constr →	$x_5 =$	0	1	1	-8
	$x_6 =$	-2	-6	0	22
	$z =$	1	4	-1	2

pivot x_5 to top using (2,2)

	x_1	x_5	x_3	1
$x_4 =$	2	4	-5	28
$x_2 =$	0	1	-1	8
$x_6 =$	-2	-6	6	-26
$z =$	1	4	-5	34

delete x_5 column:

	x_1	x_3	1
$x_4 =$	2	-5	28
$x_2 =$	0	-1	8
$x_6 =$	-2	6	-26
$z =$	1	-5	34

use (1,1) pivot to move x_1 to side.

$$
\begin{array}{r|cc|c}
 & x_4 & x_3 & 1 \\
\hline
x_1 = & \frac{1}{2} & \frac{5}{2} & -14 \\
x_2 = & 0 & -1 & 8 \\
x_6 = & -1 & 1 & 2 \\
\hline
z = & \frac{1}{2} & -\frac{5}{2} & 20
\end{array}
$$

pivot to bottom

$$
\begin{array}{r|cc|c}
 & x_4 & x_3 & 1 \\
\hline
x_2 = & 0 & -1 & 8 \\
x_6 = & -1 & 1 & 2 \\
\hline
z = & \frac{1}{2} & -\frac{5}{2} & 20 \\
\hline
x_1 = & \frac{1}{2} & \frac{5}{2} & -14
\end{array}
$$

primal feasible - proceed with simplex

pivot on (1,2)

$$
\begin{array}{r|cc|c}
 & x_4 & x_2 & 1 \\
\hline
x_3 = & 0 & -1 & 8 \\
x_6 = & -1 & -1 & 10 \\
\hline
z = & \frac{1}{2} & \frac{5}{2} & 0 \\
\hline
x_1 = & \frac{1}{2} & -\frac{5}{2} & 6
\end{array}
$$

optimal! Solution is $x = \begin{pmatrix} 6 \\ 0 \\ 8 \end{pmatrix}$

optimal objective = 0.

4.(a) write as $\min -c_1x_1 - c_2x_2 - \dots - c_nx_n$

$$\text{s.t.} \quad x_1 + x_2 + \dots + x_n = 1$$

$$x_1, x_2, \dots x_n \geq 0.$$

Dual: $\max \; u$

$$\text{s.t.} \quad u \leq -c_1$$
$$u \leq -c_2$$
$$\vdots$$
$$u \leq -c_n$$

solution is $u = -\max(c_1, c_2, \dots c_n)$.
$= \min(-c_1, -c_2, \dots -c_n)$.

~~Yes, unique!~~

(b). KKT:

$$\left.\begin{array}{l} x_1 + x_2 + \dots + x_n = 1 \\ x_1 \geq 0,\ x_2 \geq 0, \dots\ x_n \geq 0 \end{array}\right\} \text{primal feasibility}$$

$$u \leq -c_1,\ u \leq -c_2, \dots\ u \leq -c_n \ \} \text{ dual feasibility}$$

$$\left.\begin{array}{l} [u + c_1]\, x_1 = 0 \\ (u + c_2)\, x_2 = 0 \\ \vdots \\ [u + c_n]\, x_n = 0 \end{array}\right\} \text{complementarity.}$$

(c) Primal solution:

for any $i = \arg\max_k c_k$, set $x_i = 1$
and $x_j = 0$ for all $j \neq i$
easy to check that KKT is satisfied.

(d) Primal solution is unique if there is a unique index i achieving the max of the c_i's.