Midterm Examination

CS 525 - Fall 2009

Monday, October 26, 2009, 7:15-9:15pm

Each question is worth the same number of points.

No electronic computing devices, notes, or books allowed, except that you may bring one standard-size sheet of paper, handwritten on both sides, into the test. **You need to give reasoning and justify all your answers**, citing the appropriate theorems where necessary.

1. (a) For the following matrix, find the linear dependence relations between its rows and between its columns, if any.

$$A = \begin{bmatrix} 1 & 4 & 3 & -2 \\ -1 & 2 & 1 & 0 \\ -4 & 2 & 0 & 2 \end{bmatrix}.$$

   (b) What is the rank of the matrix in (a)?

   (c) Using Jordan exchanges, find the inverse of this matrix:

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}.$$

   (d) Give examples of a matrix A and right-hand sides b and c such that $Ax = b$ has no solutions while $Ax = c$ has infinitely many solutions.

2. Consider the following linear program:

$$
\begin{array}{llcrcrcrcr}
\min & & & x_1 & + & 2x_2 & + & 2x_3 & & \\
\text{subject to} & & & -2x_1 & + & x_2 & + & x_3 & \geq & 1, \\
& & & x_1 & - & x_2 & - & 2x_3 & \geq & -3, \\
& & & x_1 & , & x_2 & , & x_3 & \geq & 0.
\end{array}
$$

(a) Write down the dual of this problem.

(b) Write down the KKT conditions for this problem.

(c) Using whatever techniques you wish, find solutions to both the primal and dual.

3. By using appropriate transformations and applying Scheme II, solve the following linear program, and write down the optimal value of the objective. (Hint: You should need no more than three pivots in total.)

$$
\begin{array}{lrcrcrcr}
\max & -x_1 & - & 4x_2 & + & x_3 & - & 2 \\
\text{subject to} & 2x_1 & + & 4x_2 & - & x_3 & \geq & 4, \\
& & & x_2 & + & x_3 & = & 8, \\
& 2x_1 & + & 6x_2 & & & \leq & 22, \\
& & & & & & & \\
& x_1 & \multicolumn{4}{l}{\text{unrestricted},} & & \\
& & & x_2 & , & x_3 & \geq & 0.
\end{array}
$$

4. Consider the following linear program, where $c_1, c_2, \ldots, c_n$ are constants:

$$
\begin{aligned}
\max_{x_1, x_2, \ldots, x_n} \quad & c_1 x_1 + c_2 x_2 + c_3 x_3 + \ldots + c_n x_n \\
\text{subject to} \quad & x_1 + x_2 + x_3 + \ldots + x_n = 1, \\
& x_1, x_2, x_3, \ldots, x_n \geq 0.
\end{aligned}
$$

(a) Write down the dual of this problem, and find the solution of the dual. Is it unique?

(b) Write down the KKT conditions for this problem.

(c) Use the KKT conditions to identify a solution $(x_1, x_2, \ldots, x_n)$ to the primal.

(d) Under what condition is the primal solution unique?