525 MIDTERM SOLUTIONS 11 MARCH 2009.

1 (a)

	x_1	x_2	x_3	1
$y_1=$	1	4	7	1
$y_2=$	-1	2	1	-2

	y_1	x_2	x_3	1
$x_1=$	1	-4	-7	-1
$y_2=$	-1	6	8	-1

	y_1	y_2	x_3	1
$x_1=$	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{5}{3}$	$-\frac{5}{3}$
$x_2=$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{4}{3}$	$\frac{1}{6}$

$x_1 = -\frac{5}{3}\alpha - 5/3$

$x_2 = -4/3\,\alpha + 1/6$

$x_3 = \alpha$

$$\therefore \quad x = \begin{bmatrix} -5/3 \\ 1/6 \\ 0 \end{bmatrix} + \begin{bmatrix} -5/3 \\ -4/3 \\ 1 \end{bmatrix}\alpha, \quad \alpha \in \mathbb{R}.$$

rows are independent

(b)

	x_1	x_2	1
$y_1=$	1	3	-1
$y_2=$	2	2	1
$y_3=$	-3	-1	-4

	y_1	x_2	1
$x_1=$	1	-3	1
$y_2=$	2	-4	3
$y_3=$	-3	8	-7

	y_1	y_2	1
$x_1=$	$\frac{5}{2}$	$-\frac{3}{4}$	$-\frac{5}{4}$
$x_2=$	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{3}{4}$
$y_3=$	1	-2	-1

system is infeasible! Tableau makes no sense if $y_1=y_2=y_3=0$

dependence is (row 3) = (row 1) − 2 (row 2)

2. Convert to standard form:

$$\begin{array}{ll} \min & x_1 - 2x_2 + 2x_3 \\ \text{st} & -x_1 + x_2 - 3x_3 \geq 1 \\ & -x_1 - 4x_2 - 4x_3 \geq -2 \\ & -4x_1 - x_2 + 6x_3 \geq -5 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Construct Phase-I tableau:

	x_1	x_2	x_3	x_0	1
$x_4 =$	-1	1	-3	1	-1
$x_5 =$	-1	-4	-4	0	2
$x_6 =$	-4	-1	6	0	5
$z =$	1	-2	2	0	0
$z_0 =$	0	0	0	1	0

	x_1	x_2	x_3	x_4	1
$x_0 =$	1	-1	3	1	1
$x_5 =$	-1	-4	-4	0	2
$x_6 =$	-4	-1	6	0	5
$z =$	1	-2	2	0	0
$z_0 =$	1	-1	3	1	1

"special pivot"

	x_1	x_5	x_3	x_4	1
$x_0 =$	$\frac{5}{4}$	$\frac{1}{4}$	4	1	$\frac{1}{2}$
$x_2 =$	$-\frac{1}{4}$	$-\frac{1}{4}$	-1	0	$\frac{1}{2}$
$x_6 =$	$-\frac{15}{4}$	$\frac{1}{4}$	7	0	$\frac{9}{2}$
$z =$	$\frac{3}{2}$	$\frac{1}{2}$	4	0	-1
$z_0 =$	$\frac{5}{4}$	$\frac{1}{4}$	4	1	$\frac{1}{2}$

infeasible!
Phase I has positive optimal objective

3.

free ↓

	x_1	x_2	x_3	1
equality → $x_4 =$	2	-3	1	-4
$x_5 =$	3	-5	1	-9
$z =$	4	6	2	0

	x_1	x_2	x_4	1
$x_3 =$	-2	3	1	4
$x_5 =$	1	-2	1	-5
$z =$	0	12	2	8

	x_1	x_2	1
$x_5 =$	1	-2	-5
$z =$	0	12	8
$x_3 =$	-2	3	4

infeasible tableau - need Phase I.

	x_1	x_2	x_0	1
$x_5 =$	1	-2	1	-5
$z =$	0	12	0	8
$x_3 =$	-2	3	0	4
$z_0 =$	0	0	1	0

do special pivot

	x_1	x_2	x_5	1
$x_0 =$	-1	2	1	5
$z =$	0	12	0	8
$x_3 =$	-2	3	0	4
$z_0 =$	-1	2	1	5

↑

	x_0	x_2	x_5	1
$x_1 =$	-1	2	1	5
$z =$	0	12	0	8
$x_3 =$	2	-1	-2	-6
$z_0 =$	1	0	0	0

remove Phase I row and col.

	x_2	x_5	1
$x_1 =$	2	1	5
$z =$	12	0	8
$x_3 =$	-1	-2	-6

optimal for Phase II with solution $x = \begin{pmatrix} 5 \\ 0 \\ -6 \end{pmatrix}$

However can find other solutions by letting x_5 increase away from zero. In fact, it can increase to ∞ without going infeasible. The following ray of points is a solution:

$$x = \begin{pmatrix} 5+\alpha \\ 0 \\ -6-2\alpha \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -6 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \quad \alpha \geq 0.$$

4: (a) Problem is not infeasible because $x=0$ is a feasible point. The optimal objective is bounded above by the objective value at $x=0$, which is $\underline{0}$.

(b) Dual is

$$\max\ 0'u \quad \text{s.t.}\ A'u \le p,\quad u \ge 0.$$

Note that the objective is uniformly zero.
By strong duality, since the primal is feasible, we have two possible cases:

(i) if the dual is infeasible, then the primal is unbounded.

(ii) if the dual is feasible, then both primal and dual have solutions, and since $0'u=0$ for all u, the (common) optimal objective must be zero. Hence in this case, the point $x^*=0$ is a solution of the primal.
The necessary & sufficient condition for this to be true is dual feasibility, that is, there exists a u s.t.

$$A'u \le p,\quad u \ge 0.$$

(c) The dual has a single variable u_1:

$$\min \; u_1$$

$$\text{s.t.} \quad u_1 \geq 1$$
$$u_1 \geq 2$$
$$\vdots$$
$$u_1 \geq n$$
$$u_1 \geq 0$$

The solution is obviously $\underline{\underline{u_1 = n}}$.