Midterm Examination

CS 525 - Fall 2010

Thursday, October 28, 2010, 7:15-9:15pm

Each question is worth the same number of points.

No electronic devices, notes, or books allowed, except that you may bring one standard-size sheet of paper, handwritten on both sides, into the test. **You need to give reasoning and justify all your answers**, citing the appropriate theorems where necessary.

1. (a) For the following choice of A and b, solve the system of equations $Ax = b$ by using tableaus. If there are multiple solutions, describe the full solution set.

8

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & -1 \\ 0 & 2 & 2 \end{bmatrix}, \qquad b = \begin{bmatrix} 7 \\ -4 \\ 2 \end{bmatrix}.$$

   (b) Using Jordan exchanges, find the inverse of this matrix. If the matrix has no inverse, find a dependency relationship between its rows.

12

$$C = \begin{bmatrix} 0 & 1 & -2 \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}.$$

2. Consider the following linear program:

$$\begin{array}{lrcl} \min & 2x_1 + x_2 & & \\ & x_1 - x_2 & \geq & 4, \\ \text{subject to} & 3x_1 + 2x_2 & \geq & 10, \\ & x_1, x_2 & \geq & 0. \end{array}$$

6 (a) Write down the dual of this problem.

6 (b) Find solutions for the primal and dual.

8 (c) Suppose the right-hand side of the first constraint is changed from 4 to 10. Without performing any additional simplex iterations or referring to the tableau, give a lower bound on the optimal primal objective for the modified problem. Explain.

3. Solve the following linear program, using any of techniques we have learnt about in class (or any combination thereof). If it infeasible, say so. If it is unbounded, give a direction of unboundedness. If there are multiple solutions, describe the full set of solutions.

scheme II: 5
dual simplex: 5
solution: 4
multiple solutions: 6

$$\begin{array}{lrcl} \min & 6x_1 + 6x_2 + 2x_3 & & \\ & 4x_1 - 5x_2 + x_3 & \geq & 10, \\ \text{subject to} & 3x_1 - 3x_2 + x_3 & = & 5, \\ & x_1, x_2 & \geq & 0, \quad x_3 \text{ free.} \end{array}$$

4. Consider the following linear program:

$$\begin{aligned} \min_{x_1, x_2, \ldots, x_n} \quad & -x_1 - x_2 - x_3 - \ldots - x_n \\ \text{subject to} \quad & a_1 x_1 + a_2 x_2 + a_3 x_3 + \ldots a_n x_n \leq 1, \\ & x_1, x_2, x_3, \ldots, x_n \geq 0. \end{aligned}$$

where $a_1, a_2, \ldots, a_n$ are (strictly) positive constants.

4 (a) Write down the dual of this problem.

4 (b) Find the solution of the dual, by inspection. Is this solution unique?

4 (c) Write down the KKT conditions for this problem.

4 (d) By using the KKT conditions (or by any other means), find a solution of the original (primal) linear program. (It if sufficient to write down just one solution.)

4 (e) How do the primal and dual solutions change if one or more of the constants a_i, $i = 1, 2, \ldots, n$ is allowed to be zero?

525 MIDTERM FALL 2010

1 (a).

$$
\begin{array}{r|ccc|c|}
 & x_1 & x_2 & x_3 & 1 \\
\hline
y_1 = & 1 & 3 & 4 & -7 \\
y_2 = & 2 & 6 & -1 & 4 \\
y_3 = & 0 & 2 & 2 & -2 \\
\hline
\end{array}
\longrightarrow
\begin{array}{r|ccc|c|}
 & y_1 & x_2 & x_3 & 1 \\
\hline
x_1 = & 1 & -3 & -4 & 7 \\
y_2 = & 2 & 0 & -9 & 18 \\
y_3 = & 0 & 2 & 2 & -2 \\
\hline
\end{array}
$$

$$\downarrow$$

$$
\begin{array}{r|ccc|c|}
 & y_1 & y_2 & y_3 & 1 \\
\hline
x_1 = & & \frac{1}{9} & & 2 \\
x_2 = & \frac{-2}{9} & \frac{1}{9} & \frac{1}{2} & -1 \\
x_3 = & & \frac{-1}{9} & & 2 \\
\hline
\end{array}
\longleftarrow
\begin{array}{r|ccc|c|}
 & y_1 & x_2 & y_3 & 1 \\
\hline
x_1 = & 1 & 1 & -2 & 3 \\
y_2 = & 2 & 9 & \frac{-9}{2} & 9 \\
x_3 = & 0 & -1 & \frac{1}{2} & 1 \\
\hline
\end{array}
$$

solution $x = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

(b)

$$
\begin{array}{r|ccc|}
 & x_1 & x_2 & x_3 \\
\hline
y_1 = & 0 & 1 & -2 \\
y_2 = & 2 & 3 & -1 \\
y_3 = & 1 & -1 & 3 \\
\hline
\end{array}
\longrightarrow
\begin{array}{r|ccc|}
 & x_1 & y_1 & x_3 \\
\hline
x_2 = & 0 & 1 & 2 \\
y_2 = & 2 & 3 & 5 \\
y_3 = & 1 & -1 & 1 \\
\hline
\end{array}
$$

$$\downarrow$$

$$
\begin{array}{r|ccc|}
 & y_3 & y_1 & y_2 \\
\hline
x_2 = & \frac{-4}{3} & \frac{-7}{3} & \frac{2}{3} \\
x_3 = & \frac{-2}{3} & \frac{-5}{3} & \frac{1}{3} \\
x_1 = & \frac{5}{3} & \frac{8}{3} & \frac{-1}{3} \\
\hline
\end{array}
\longleftarrow
\begin{array}{r|ccc|}
 & y_3 & y_1 & x_3 \\
\hline
x_2 = & 0 & 1 & 2 \\
y_2 = & 2 & 5 & 3 \\
x_1 = & 1 & 1 & -1 \\
\hline
\end{array}
$$

$\downarrow$ permute columns

$$
\begin{array}{r|ccc|}
 & y_1 & y_2 & y_3 \\
\hline
x_2 = & -\frac{7}{3} & 2/3 & -4/3 \\
x_3 = & -\frac{5}{3} & 1/3 & -2/3 \\
x_1 = & 8/3 & -1/3 & 5/3 \\
\hline
\end{array}
\xrightarrow{\text{permute rows}}
\begin{array}{r|ccc|}
 & y_1 & y_2 & y_3 \\
\hline
x_1 = & 8/3 & -1/3 & 5/3 \\
x_2 = & -7/3 & 2/3 & -4/3 \\
x_3 = & -5/3 & 1/3 & -2/3 \\
\hline
\end{array}
$$

$$
C^{-1} = \begin{bmatrix} \frac{8}{3} & -\frac{1}{3} & \frac{5}{3} \\ -\frac{7}{3} & \frac{2}{3} & -\frac{4}{3} \\ -\frac{5}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}.
$$

2(a) $\max \ 4u_1 + 10u_2$

s.t. $u_1 + 3u_2 \le 2$

$-u_1 + 2u_2 \le 1$

$u_1, u_2 \ge 0$

(b)

		$u_3 =$ x_1	$u_4 =$ x_2	$w =$ 1
$-u_1$	$x_3 =$	1	-1	-4
$-u_2$	$x_4 =$	3	2	-10
1	$z =$	2	1	0

$\downarrow$ dual simplex chooses (1,1) pivot

		$u_1 =$ x_3	$u_4 =$ x_2	$w =$ 1
$-u_3$	$x_1 =$	1	1	4
$-u_2$	$x_4 =$	3	5	2
1	$z =$	2	3	8

solution: primal $x^* = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$ dual $u^* = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

(c). changing b_1 from 4 to 10 does not change dual constraints, so u^* is feasible for the modified dual, with objective $10u_1^* + 10u_2^* = 20$. By weak duality, this is a lower bound on the optimal objective of the modified primal

(d) ~~x^* is still feasible for the modified primal, and u^* is still feasible for the modified dual.~~

~~The modified primal objective at x^* is still 8~~

~~The modified dual objective at u^* is still 8~~

~~Hence by strong duality, x^* solves modified primal and u^* solves modified dual.~~

3

	x_1	x_2	x_3	1
$x_4 =$	4	-5	1	-10
$x_5 =$	3	-3	1	-5
$z =$	6	6	2	0

scheme II

	x_1	x_2	$\cancel{x_5}$	1
$x_4 =$	1	-2	1	-5
$x_3 =$	-3	3	1	5
$z =$	0	12	2	10

remove $x_5 = 0$

move x_3 to bottom

	x_1	x_2	1
$x_4 =$	1	-2	-5
$z =$	0	12	10
$x_3 =$	-3	-3	5

dual simplex

pivot on (1,1)

	x_4	x_2	1
$x_1 =$	1	2	5
$z =$	0	12	10
$x_3 =$	-3	-3	-10

solution: $x = \begin{pmatrix} 5 \\ 0 \\ -10 \end{pmatrix}$

non unique can get additional solutions by letting $x_4 = \alpha \uparrow \infty$

$$x = \begin{pmatrix} 5 \\ 0 \\ -10 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \alpha$$

4. (a) First change to standard form:

$$\min \; -x_1 - x_2 \cdots - x_n$$
$$\text{s.t.} \quad -a_1x_1 - a_2x_2 - \cdots - a_nx_2 \geq -1$$
$$x_1, x_2 \ldots x_n \geq 0$$

Dual is

$$\max_u \; -u$$
$$\text{s.t.} \quad -a_iu \leq -1, \quad i = 1 \ldots n$$
$$u \geq 0$$

or equivalently

$$\min \; u$$
$$\text{s.t.} \quad a_iu \geq 1 \quad i = 1 \ldots n$$
$$u \geq 0$$

(b) solution is $u^* = \dfrac{1}{\min_{i=1\ldots n}(a_i)}$

yes, it is unique

(c) KKT: $0 \leq x_i \perp -1 + a_iu \geq 0 \quad i = 1 \ldots n.$

$$0 \leq 1 - a_1x_1 - \cdots - a_nx_n \perp u \geq 0$$

(d). Let $\mathcal{J} = \{ j \mid j \in \arg\min_i a_i \}$, that is, $\mathcal{J}$ is the set of indices j such that a_j is a minimizer of $\{a_1, a_2, \ldots a_n\}$

then for $j \notin \mathcal{J}$ we have $-1 + a_ju^* > 0 \Rightarrow x_j^* = 0$

while for $j \in \mathcal{J}$ we may have $x_j^* > 0$.

Since $u^* > 0$, we have $\sum_{i=1}^{n} a_ix_i^* = \sum_{i \in \mathcal{J}} a_ix_i^* = 1$.

These conditions together define x^*.

One such solution would be: for some $j \in \mathcal{J}$, $x_j^* = \frac{1}{a_j}$, $x_i^* = 0, \; i \neq j$

(e) if any $a_i = 0$, the dual is infeasible, while the primal is clearly still feasible ($x = 0$ is a feasible point). Hence, by strong duality the primal is unbounded. (we can also see this by inspection: for any i such that $a_i = 0$, we can let $x_i \uparrow \infty$ without affecting feasibility, while driving the objective to $-\infty$.)