CS525 MIDTERM 10/30/12 FALL 2012

(1) (a)

$$
\begin{array}{c|cccc|c}
 & x_1 & x_2 & x_3 & x_4 & 1 \\
\hline
y_1 = & 0 & -1 & 3 & 3 & -1 \\
y_2 = & 2 & 5 & -1 & 0 & 1 \\
y_3 = & 2 & 2 & 8 & 9 & -2
\end{array}
$$

$\downarrow$

$$
\begin{array}{c|cccc|c}
 & x_1 & y_1 & x_3 & x_4 & 1 \\
\hline
x_2 = & 0 & -1 & 3 & 3 & -1 \\
y_2 = & 2 & -5 & 14 & 15 & -4 \\
y_3 = & 2 & -2 & 14 & 15 & -4
\end{array}
$$

$\downarrow$

$$
\begin{array}{c|cccc|c}
 & y_2 & y_1 & x_3 & x_4 & 1 \\
\hline
x_2 = & 0 & -1 & 3 & 3 & -1 \\
x_1 = & \frac{1}{2} & \frac{5}{2} & -7 & -\frac{15}{2} & 2 \\
y_3 = & 1 & 3 & 0 & 0 & 0
\end{array}
$$

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Set $y_1 = y_2 = y_3 = 0 \Rightarrow$ system is consistent.
Multiple solutions satisfy:

$$x_2 = 3x_3 + 3x_4 - 1$$
$$x_1 = -7x_3 - \tfrac{15}{2}x_4 + 2$$

Set $x_3 = \alpha$, $x_4 = \beta$, to obtain solution set

$$
\left\{ \begin{bmatrix} -7\alpha - \frac{15}{2}\beta + 2 \\ 3\alpha + 3\beta - 1 \\ \alpha \\ \beta \end{bmatrix} : \begin{array}{l} \alpha \in \mathbb{R} \\ \beta \in \mathbb{R} \end{array} \right\}
$$

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1 (b) Rows of A are linearly dependent and satisfy

5 $$A_{3\cdot} = A_{2\cdot} + 3A_{1\cdot}$$

(2) $\min c^T x$ st $Ax \geq b$, $x \geq 0$.

4 (a) dual is $\max b^T u$ st. $A^T u \leq c$, $u \geq 0$.

3 Dual has a solution u^* by strong duality

8 (b) When c is replaced by $3c$, solution x^* of primal remains the same.

The modified dual has solution $\bar{u} = 3u^*$.

To verify, note that $\bar{u}$ is feasible for the modified dual, since

$$A^T\bar{u} = 3A^T u^* \leq 3c \quad ✓$$

$$\bar{u} = 3u^* \geq 0 \quad ✓$$

At $\bar{u}$, modified dual objective is

$$b^T\bar{u} = 3b^T u^* = 3c^T x^* = \text{modified primal optimal objective}$$

Hence $\bar{u}$ is feasible for the modified dual and attains the same objective value as the optimal objective for the modified primal. Hence by duality (weak or strong) it follows that $\bar{u} = 3u^*$ is optimal for modified dual.

5 (c) With this modification, primal could have a solution ~~x~~ be unbounded.

3 (a) Dual: $\max\ 5u_1 + 6u_2$

$$\text{s.t.}\quad u_1 + 3u_2 \le 2$$
$$-2u_1 + u_2 = -1$$
$$u_1 \ge 0.$$

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(b) KKT:

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$$0 \le x_1 \perp u_1 + 3u_2 - 2 \le 0$$
$$-2u_1 + u_2 = -1$$
$$0 \le u_1 \perp x_1 - 2x_2 - 5 \ge 0$$
$$3x_1 + x_2 = 6$$

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(c) From KKT conditions $x_1^* > 0 \Rightarrow$

$$u_1 + 3u_2 = 2$$

while we also have from KKT conditions that

$$-2u_1 + u_2 = -1.$$

By solving these two equations for u_1, u_2 we get

$$u_2 = 3/7$$
$$u_1 = 5/7.$$

(Checking the other KKT conditions we see that

$0 < u_1 \perp x_1 - 2x_2 - 5 = 0$ ✓

$3x_1 + x_2 = \frac{51}{7} - \frac{9}{7} = \frac{42}{7} = 6$ ✓)

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(4) (a)

	x_1	x_2	x_3	1
$x_4 =$	4	-1	1	-10
$x_5 =$	1	1	1	-5
$z =$	2	0	2	0

Need $x_5 = 0$ for equality constraint

Scheme I pivot

	~~x_5~~	x_2	x_3	1
$x_4 =$	4	-5	-3	10
$x_1 =$	1	-1	-1	5
$z =$	2	-2	0	10

	x_2	x_3	1
$x_4 =$	-5	-3	10
$x_1 =$	-1	-1	5
$z =$	-2	0	10

↑

	x_4	x_3	1
$x_2 =$	$\frac{-1}{5}$	$\frac{-3}{5}$	2
$x_1 =$	$\frac{1}{5}$	$\frac{-2}{5}$	3
$z =$	$\frac{2}{5}$	$\frac{6}{5}$	6

Solution $x_1 = 3$

$x_2 = 2$

$x_3 = 0$

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(4) (6)

	x_1	x_2	x_3	1
$x_4 =$	1	−1	3	−10
$x_5 =$	1	0	1	−5
$z =$	2	3	2	0

dual simplex

	x_1	x_2	x_4	1
$x_3 =$	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{10}{3}$
$x_5 =$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{5}{3}$
$z =$	$\frac{4}{3}$	$\frac{11}{3}$	$\frac{2}{3}$	$\frac{20}{3}$

	x_5	x_2	x_4	1
$x_3 =$	$-\frac{1}{2}$			$\frac{5}{2}$
$x_1 =$	$\frac{3}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{5}{2}$
$z =$	2	3	0	10

$\frac{10}{3} - \frac{5/9}{2/3} = \frac{20}{6} - \frac{5}{6} = \frac{15}{6}$

$= \frac{5}{2}$

optimal

solution: $x_1 = 5/2$

$x_2 = 0$

$x_3 = 5/2$

check: $z = 2x_1 + 3x_2 + 2x_3 = 10$ ✓

$x_1 - x_2 + 3x_3 = \frac{5}{2} - 0 + \frac{15}{2} = 10$ ✓

$x_1 + x_3 = 5$ ✓