Midterm Examination

CS 525 - Fall 2012

Tuesday, October 30, 2012, 7:15pm–9:15pm

Each of the four questions is worth the same number of points.

No electronic devices, notes, or books allowed, except that you may bring one standard-size sheet of paper, handwritten on both sides, into the test. **You need to give reasoning and justify all your answers**, citing the appropriate theorems where necessary.

1. For the following choice of A and b, solve the system of equations $Ax = b$ by using tableaus.

$$A = \begin{bmatrix} 0 & -1 & 3 & 3 \\ 2 & 5 & -1 & 0 \\ 2 & 2 & 8 & 9 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

   (a) If there are multiple solutions, describe the full solution set. If there are no solutions, say so. If there is a unique solution, write it down.

   (b) Are the rows of A linearly independent or linearly dependent? If linearly dependent, write down the relationship between the rows of A.

2. Suppose that the following linear program has a solution x^*:

$$\min_x \; c^T x \;\text{ subject to }\; Ax \geq b, \;\; x \geq 0.$$

   (a) Write down the dual of this problem and explain why there exists a solution u^* to the dual.

(b) Suppose that the cost vector is tripled in the linear program, that is, c is replaced by $3c$ in the formulation above. For this modified problem give solutions to both the *primal* and the *dual*. (Explain your answers.)

(c) Suppose that we *negate* the cost vector, that is, we replace c by $-c$ in the formulation above, and thus seek the minimizer of $-c^T x$ over the original feasible set. What can you say about the solutions of the modified primal and dual problems?

3. Consider the following linear program, in general form:

$$\begin{array}{lrcl} \min & & 2x_1 - x_2 & \\ & x_1 - 2x_2 & \geq & 5, \\ \text{subject to} & 3x_1 + x_2 & = & 6, \\ & x_1 & \geq & 0, \quad x_2 \text{ free.} \end{array}$$

(a) Write down the dual of this problem.

(b) Write down the KKT conditions for this problem.

(c) The solution of this problem is $x^* = (\frac{17}{7}, \frac{-9}{7})'$. Using the KKT conditions, or by any other means, find a solution of the dual.

4. Solve the following linear programs, using any of the techniques we have learnt about in class. Show your working for each. (Each question can be solved using at most two Jordan exchanges.)

(a)

$$\begin{array}{lrcl} \min & & 2x_1 + 2x_3 & \\ & 4x_1 - x_2 + x_3 & \geq & 10, \\ \text{subject to} & x_1 + x_2 + x_3 & = & 5, \\ & x_1, x_2, x_3 & \geq & 0. \end{array}$$

(Note that the second constraint is an equality.)

(b)

$$\begin{array}{lrcl} \min & & 2x_1 + 3x_2 + 2x_3 & \\ & x_1 - x_2 + 3x_3 & \geq & 10, \\ \text{subject to} & x_1 + x_3 & \geq & 5, \\ & x_1, x_2, x_3 & \geq & 0. \end{array}$$