525 MIDTERM FALL 2014

(1) (a)

$$\begin{array}{c|cccc|c} & x_1 & x_2 & x_3 & x_4 & 1 \\ \hline y_1 = & -1 & 0 & 3 & 4 & -3 \\ y_2 = & 2 & 1 & -1 & 6 & -1 \\ y_3 = & 4 & 1 & -7 & -2 & 5 \end{array}$$

$\downarrow$

$$\begin{array}{c|cccc|c} & y_1 & x_2 & x_3 & x_4 & 1 \\ \hline x_1 = & -1 & 0 & 3 & 4 & -3 \\ y_2 = & -2 & 1 & 5 & 14 & -7 \\ y_3 = & -4 & 1 & 5 & 14 & -7 \end{array}$$

$\downarrow$

$$\begin{array}{c|cccc|c} & y_1 & y_2 & x_3 & x_4 & 1 \\ \hline x_1 = & -1 & 0 & 3 & 4 & -3 \\ x_2 = & 2 & 1 & -5 & -14 & 7 \\ y_3 = & -2 & 1 & 0 & 0 & 0 \end{array}$$

set $y=0$, choose $x_3=\alpha$, $x_4=\beta$, obtain solutions

$$x_1 = 3\alpha + 4\beta - 3$$
$$x_2 = -5\alpha - 14\beta + 7$$
$$x_3 = \alpha$$
$$x_4 = \beta$$

Solution set:

$$\left\{ \begin{pmatrix} -3 \\ 7 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ -5 \\ 1 \\ 0 \end{pmatrix}\alpha + \begin{pmatrix} 4 \\ -14 \\ 0 \\ 1 \end{pmatrix}\beta : \begin{matrix} \alpha \in \mathbb{R} \\ \beta \in \mathbb{R} \end{matrix} \right\}$$

(b) linearly dependent: $A_{3\cdot} = -2A_{1\cdot} + A_{2\cdot}$

(c) $\left\{ \begin{pmatrix} -3 \\ 7 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ -5 \\ 1 \\ 0 \end{pmatrix} \alpha + \begin{pmatrix} 4 \\ -14 \\ 0 \\ 1 \end{pmatrix} \beta : \begin{array}{l} 7 - 5\alpha - 14\beta \geq 5 \\ \alpha \in \mathbb{R}, \beta \in \mathbb{R} \end{array} \right\}$

(d) Solution of (a) is a (two-dimensional) plane
Solution of (d) is a (two-dimensional) half-plane

(2) This problem is clearly infeasible, since the constraint $-x_2 - x_3 \geq 3$ cannot hold when $x_2 \geq 0$ and $x_3 \geq 0$.

We can see this too by setting up the tableau with dual labels:

		$u_3 =$	$u_4 =$	$u_5 =$	$w =$
		x_1	x_2	x_3	1
$-u_1$	$x_4 =$	-1	2	6	-5
$-u_2$	$x_5 =$	0	-1	-1	-3
1	$z =$	0	2	5	0

The tableau is dual feasible, so we can proceed to apply dual simplex without a "Phase I".
But if we select row 2 as a pivot row, we see that there is no suitable pivot column - the dual variable u_2 can go to $+\infty$ while remaining dual feasible and driving the dual objective to $+\infty$. Hence the dual is unbounded, so by strong duality, the primal is infeasible

(3) (a) First write the primal in "general" form.

$$\min \quad x_1 + x_2 - x_3$$
$$\text{s.t.} \quad -x_1 - x_2 \geq -1$$
$$5x_2 + x_3 \geq 3$$
$$x_1 + x_2 + x_3 = -1$$
$$x_1 \geq 0, \; x_2 \geq 0 \quad (x_3 \text{ free})$$

Dual is

$$\max \quad -u_1 + 3u_2 - u_3$$
$$\text{s.t.} \quad -u_1 + u_3 \leq 1$$
$$-u_1 + 5u_2 + u_3 \leq 1$$
$$u_2 + u_3 = -1$$
$$u_1 \geq 0, \; u_2 \geq 0 \quad (u_3 \text{ free})$$

. Feasible point: $u = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ giving objective 1

(b) KKT:

$$0 \leq -x_1 - x_2 + 1 \perp u_1 \geq 0$$
$$0 \leq 5x_2 + x_3 - 3 \perp u_2 \geq 0$$
$$x_1 + x_2 + x_3 = -1$$
$$0 \leq u_1 - u_3 + 1 \perp x_1 \geq 0$$
$$0 \leq u_1 - 5u_2 - u_3 + 1 \perp x_2 \geq 0$$
$$u_2 + u_3 = -1$$

(c) Lower bound on primal obj given by value of dual objective at a dual feasible point (by weak duality). The point $u = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ gives a dual objective of 1, so this is a lower bound

(d) Start with the general form of the primal:

	x_1	x_2	x_3	1
$x_4 =$	-1	-1	0	1
$x_5 =$	0	5	1	-3
$x_6 =$	1	1	1	1
$z =$	1	1	-1	0

Scheme II

	x_1	x_2	x_6	1
$x_4 =$	-1	-1	0	1
$x_5 =$	-1	4	1	-4
$x_3 =$	-1	-1	1	-1
$z =$	2	2	-1	1

delete x_6 column (equality constraint)

	x_1	x_2	1
$x_4 =$	-1	-1	1
$x_5 =$	-1	4	-4
$x_3 =$	-1	-1	-1
$z =$	2	2	1

move x_3 to bottom (free variable)

	x_1	x_2	1
$x_4 =$	-1	-1	1
$x_5 =$	-1	4	-4
$z =$	2	2	1
$x_3 =$	-1	-1	-1

Dual Simplex:

	x_1	x_5	1
$x_4 =$	$-5/4$	$-1/4$	0
$x_2 =$	$1/4$	$1/4$	1
$z =$	$5/2$	$1/2$	3
$x_3 =$	$-5/4$	$-1/4$	-2

optimal!

Solution is $x^* = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$ with optimal objective 3

(c) when we plug x^* into the KKT conditions, we find that the following conditions on u must be satisfied.

$$u_1 \geq 0$$
$$u_2 \geq 0$$
$$u_1 - u_2 + 1 \geq 0$$
$$u_1 - 5u_2 - u_3 + 1 = 0$$
$$u_2 + u_3 = -1$$

Solve these for u_1, u_2, u_3.

Setting $u_1 = 0$, we need u_2 and u_3 to satisfy:

$$u_2 \geq 0$$
$$-u_2 + 1 \geq 0$$
$$-5u_2 - u_3 = 1 - 1$$
$$u_2 + u_3 = -1$$

solving the last two equations, we obtain.

$$-4u_2 = -2 \Rightarrow u_2 = 1/2$$
$$\Rightarrow u_3 = -1 - 1/2 = -3/2$$

thus a dual solution is $u = \begin{pmatrix} 0 \\ 1/2 \\ -3/2 \end{pmatrix}$ with dual objective 3.

(4) (a) Problem can't become infeasible because its constraints don't change, and the original form is known to be feasible

Yes, it is possible to choose $\hat{p}$ so that the problem has a solution. e.g. set $\hat{p}=0$, then the solution is any feasible point

(b) Writing the dual of the LP as

$$\max b^T u$$
$$\text{st } A^T u \le \hat{p} \quad , u \ge 0$$

we see that $\hat{u}$ satisfies $A^T\hat{u} \le p = A^T\hat{u}+e$, so that $\hat{u}$ is dual feasible. Since we know that the primal is also feasible, strong duality tells us that both primal and dual have a solution

(c) When we set $b=0$ the problem becomes

P' $\min p^T x$ st $Ax \ge 0$, $x \ge 0$

while its dual becomes

D' $\max 0^T u$ st $A^T u \le p$, $u \ge 0$

Since the original primal was unbounded, its dual must be infeasible, by strong duality. But the constraints of D' are the same as for the original dual, so D' is also infeasible. Hence P' is either infeasible or unbounded (strong duality). Since $x=0$ is a feasible point for P', it must be unbounded