Midterm Examination

CS 525 - Fall 2014

Monday, October 27, 2014, 7:15-9:15pm

No electronic devices, notes, or books allowed, except that you may bring one standard-size sheet of paper, handwritten on both sides, into the test. **You need to give reasoning and justify all your answers**, citing the appropriate theorems where necessary.

1. Consider the following matrix and vector:

$$A = \begin{bmatrix} -1 & 0 & 3 & 4 \\ 2 & 1 & -1 & 6 \\ 4 & 1 & -7 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}.$$

   (a) Find *all* solutions to the linear system of equations $Ax = b$.

   (b) Are the rows of the matrix A linearly independent or linearly dependent? If linearly dependent, state the linear dependence relationship between them.

   (c) Suppose that in addition to $Ax = b$, we impose the additional constraint on the solution that $x_2 \geq 5$. By modifying your solution to (a), write down the solution set of this expanded system.

   (d) In geometric terms, what kinds of shapes do your solution sets in (a) and (c) represent? (A line, a ray, a plane, a point, or some other shape?)

2. Consider the following linear program in standard form:

$$\begin{aligned} \min_{x_1,x_2,x_3} \quad & 2x_2 + 5x_3 \\ \text{subject to} \quad & -x_1 + 2x_2 + 6x_3 \geq 5, \\ & -x_2 - x_3 \geq 3, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

Find *all* solutions of this problem. Say which are the *vertex* solutions.

3. Consider the following linear program:

$$\begin{aligned} \min_{x_1,x_2,x_3} \quad & x_1 + x_2 - x_3 \\ \text{subject to} \quad & x_1 + x_2 \le 1, \\ & 5x_2 + x_3 \ge 3, \\ & x_1 + x_2 + x_3 = -1, \\ & x_1, x_2 \ge 0. \quad (x_3 \text{ free}) \end{aligned}$$

   (a) Write down the dual of this problem and find (by "eyeballing") a feasible point for the dual.

   (b) Write down the KKT (optimality) conditions for this problem.

   (c) Without constructing any tableaus, find a lower bound on the optimal objective value for the linear program above.

   (d) Solve the problem above. (Hint: You might find Scheme II and the dual simplex method helpful.)

   (e) Use the KKT conditions to construct a solution to the dual.

4. Consider the following linear program in standard form:

$$\min_x \; p^T x \;\text{ subject to }\; Ax \ge b, \;\; x \ge 0. \tag{1}$$

   Suppose that this problem is an *unbounded* linear program.

   (a) If we change p to some other vector $\tilde{p}$, could the problem become *infeasible* rather than unbounded? Is it possible to choose $\tilde{p}$ so that the problem *has a solution*?

   (b) Define $\hat{p} = A^T\hat{u} + e$, where $e = (1, 1, \ldots, 1)^T$ and $\hat{u}$ is some given vector with nonnnegative entries. If we replace p in (1) by the vector $\hat{p}$ defined here, can we guarantee that this modified linear program is unbounded, that is is infeasible, or that it has a solution? Or can we not say for sure whether it is in any one of these three categories?

   (c) Suppose we take the unbounded linear program (1) and replace b by the zero vector. Is the modified problem unbounded, infeasible, or does it have a solution? Or can we not say for sure if it is any of these?