525 MIDTERM SPRING 2014

(1).

$$\begin{array}{c|ccc|c} & x_1 & x_2 & x_3 & 1 \\ \hline x_4 = & 2 & -3 & 0 & -4 \\ x_5 = & 3 & -5 & 1 & -9 \\ \hline z = & 2 & 3 & 1 & 0 \end{array}$$

Do Scheme II

$\downarrow$

$$\begin{array}{c|ccc|c} & \cancel{x_4} & x_2 & x_3 & 1 \\ \hline x_1 = & \frac{1}{2} & \frac{3}{2} & 0 & 2 \\ x_5 = & \frac{3}{2} & -\frac{1}{2} & 1 & -3 \\ \hline z = & 1 & 6 & 1 & 4 \end{array}$$

$\downarrow$

$$\begin{array}{c|cc|c} & x_2 & x_3 & 1 \\ \hline x_1 = & \frac{3}{2} & 0 & 2 \\ x_5 = & -\frac{1}{2} & 1 & -3 \\ \hline z = & 6 & 1 & 4 \end{array}$$

$\downarrow$

$$\begin{array}{c|cc|c} & x_2 & x_5 & 1 \\ \hline x_1 = & 3/2 & 0 & 2 \\ x_3 = & 1/2 & 1 & 3 \\ \hline & 13/2 & 1 & 7 \end{array}$$

Optimal tableau! No further pivots needed.

Solution. $x = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$

(2) (a)

$$\begin{array}{c|ccc|c} & x_1 & x_2 & x_3 & 1 \\ \hline y_1 = & 1 & -3 & 0 & 2 \\ y_2 = & 2 & 4 & 1 & -3 \\ y_3 = & 0 & -10 & -1 & 7 \end{array}$$

$\downarrow$

$$\begin{array}{c|ccc|c} & y_1 & x_2 & x_3 & 1 \\ \hline x_1 = & 1 & 3 & 0 & -2 \\ y_2 = & 2 & 10 & 1 & -7 \\ y_3 = & 0 & -10 & -1 & 7 \end{array}$$

$\downarrow$

$$\begin{array}{c|ccc|c} & y_1 & x_2 & y_3 & 1 \\ \hline x_1 = & 1 & 3 & 0 & -2 \\ y_2 = & 2 & 0 & -1 & 0 \\ x_3 = & 0 & -10 & -1 & 7 \end{array}$$

cannot pivot y_2 to top, so the matrix has dependency between rows.

(row 2) = 2 × (row 1) − (row 3)

But linear system has a solution. Can choose $x_2 = \lambda$ to be any value, and set

$x_1 = 3\lambda - 2$

$x_3 = -10\lambda + 7.$

that is,

$$x = \begin{pmatrix} 3\lambda - 2 \\ \lambda \\ -10\lambda + 7 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 7 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ -10 \end{pmatrix} \lambda, \quad \forall \lambda \in \mathbb{R}$$

(2)(b)

$$\begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline y_1 = & 0 & 1 & -3 \\ y_2 = & 1 & -1 & 2 \\ y_3 = & -2 & 3 & 0 \end{array}$$

$$\downarrow$$

$$\begin{array}{c|ccc} & x_1 & y_2 & x_3 \\ \hline y_1 = & 1 & -1 & -1 \\ x_2 = & 1 & -1 & 2 \\ y_3 = & 1 & -3 & 6 \end{array}$$

$$\downarrow$$

$$\begin{array}{c|ccc} & y_1 & y_2 & x_3 \\ \hline x_1 = & 1 & 1 & 1 \\ x_2 = & 1 & 0 & 3 \\ y_3 = & 1 & -2 & 7 \end{array}$$

$$\downarrow$$

$$\begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 = & \frac{6}{7} & \frac{9}{7} & \frac{1}{7} \\ x_2 = & \frac{4}{7} & \frac{6}{7} & \frac{3}{7} \\ x_3 = & -\frac{1}{7} & \frac{2}{7} & \frac{1}{7} \end{array}$$

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 6 & 9 & 1 \\ 4 & 6 & 3 \\ -1 & 2 & 1 \end{bmatrix}$$

(3)

	x_1	x_2	x_3	1
$x_4 =$	1	1	1	-2
$x_5 =$	2	-5	1	1
$z =$	2	3	-7	0

Phase I:

	x_1	x_2	x_3	x_0	1
$x_4 =$	1	1	1	1	-2
$x_5 =$	2	-5	1	0	1
$z =$	2	3	-7	0	0
$z_0 =$	0	0	0	1	0

Special Pivot

	x_1	x_2	x_3	x_4	1
$x_0 =$	-1	-1	-1	1	2
$x_5 =$	2	-5	1	0	1
$z =$	2	3	-7	0	0
$z_0 =$	-1	-1	-1	1	2

	x_0	x_2	x_3	x_4	1
$x_1 =$	-1	-1	-1	1	2
$x_5 =$	-2	-7	-1	2	5
$z =$	-2	1	-9	2	4
~~$z_0 =$~~	~~1~~	~~0~~	~~0~~	~~0~~	~~0~~

Phase I complete

	x_2	x_3	x_4	1
$x_1 =$	-1	-1	1	2
$x_5 =$	-7	-1	2	5
$z =$	1	-9	2	4

↓

$$\begin{array}{c|ccc|c} & x_2 & x_1 & x_4 & 1 \\ \hline x_3 = & -1 & -1 & 1 & 2 \\ x_5 = & -6 & 1 & 1 & 3 \\ \hline z = & 10 & 9 & -7 & -14 \end{array}$$

unbounded! Set $x_4 = \lambda > 0$, then

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2+\lambda \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}}_{u} + \lambda \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{v}$$

yields $z = -14 - 7\lambda$

which approaches $-\infty$ as $\lambda \uparrow \infty$

(4) (a) Dual:

$$\max \quad u_1 + u_2$$
$$\text{s.t.} \quad a_1 u_1 + u_2 \le 1$$
$$a_2 u_1 + u_2 \le 2$$
$$a_2 u_1 + u_2 \le 3$$
$$\vdots$$
$$a_n u_1 + u_2 \le n.$$
$$u_1 \ge 0, \; u_2 \ge 0.$$

(b). Because $a_1 > a_2$ and $u_1 \ge 0$, we have

$$a_2 u_1 + u_2 \le a_1 u_1 + u_2 \le 1 < 2$$

that is, the second constraint must be inactive if the first constraint is satisfied! The same is true of all other constraints! Hence only the first constraint can be active

(c) Because all but the first constraint are redundant, the dual problem is equivalent to the following:

$$\max \quad u_1 + u_2$$
$$\text{s.t.} \quad a_1 u_1 + u_2 \le 1$$
$$u_1 \ge 0, \quad u_2 \ge 0.$$

The vertices of the feasible region, obtained by considering different pairs of active constraints are

$$u = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; u = \begin{pmatrix} 1/a_1 \\ 0 \end{pmatrix}$$

Since $a_1 \geq 1$, we have $1/a_1 < 1$.

So the vertex with the best objective is

$$u^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

this is the dual solution

(d) KKT:

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n \geq 1$$
$$x_1 + x_2 + \dots + x_n \geq 1$$
$$x_1 \geq 0, x_2 \geq 0, \dots x_n \geq 0$$
$$a_1 u_1 + u_2 \leq 1$$
$$a_2 u_1 + u_2 \leq 1$$
$$\vdots$$
$$a_n u_1 + u_n \leq 1$$
$$u_1 \geq 0, u_2 \geq 0.$$
$$u_1 [a_1 x_1 + a_2 x_2 + \dots + a_n x_n - 1] = 0$$
$$u_2 [x_1 + x_2 + \dots + x_n - 1] = 0$$
$$x_1 [a_1 u_1 + u_2 - 1] = 0$$
$$x_2 [a_2 u_1 + u_2 - 1] = 0$$
$$\vdots$$
$$x_n [a_n u_1 + u_2 - 1] = 0$$

(e) Since all but the first dual constraint are inactive, we have $x_2 = x_3 = \dots = x_n = 0$.

Since $u_2 > 0$, the second primal constraint is active, and we have

$$x_1 + x_2 \dots + x_n - 1 = 0 \Rightarrow x_1 - 1 = 0 \Rightarrow x_1 = 1$$

So primal solution is $x^* = [1, 0, 0 \dots, 0]^T$.