Midterm Examination

CS 525 - Spring 2014

Wednesday, March 12, 2014, 7:15-9:15pm

No electronic devices, notes, or books allowed, except that you may bring one standard-size sheet of paper, handwritten on both sides, into the test. **You need to give reasoning and justify all your answers**, citing the appropriate theorems where necessary.

1. (10 points) Solve the following linear program. (Hint: Use Scheme II.) If it infeasible, say so. If it is unbounded, give a direction of unboundedness. If there are multiple solutions, describe the full set of solutions.

$$\begin{array}{lrcl} \min & \multicolumn{3}{c}{2x_1 + 3x_2 + x_3} \\ & 2x_1 - 3x_2 & = & 4, \\ \text{subject to} & 3x_1 - 5x_2 + x_3 & \geq & 9, \\ & x_1, x_2 & \geq & 0, \quad x_3 \text{ free.} \end{array}$$

2. (10 points)

   (a) Find all solutions of the linear system $Ax = b$, where A and b are given below. If the rows of A are linearly dependent, write out the dependence relation.

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 2 & 4 & 1 \\ 0 & -10 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} -2 \\ 3 \\ -7 \end{bmatrix}.$$

   (b) Using Jordan exchanges, find the inverse of the following matrix:

$$A = \begin{bmatrix} 0 & 1 & -3 \\ 1 & -1 & 2 \\ -2 & 3 & 0 \end{bmatrix}.$$

3. (10 points) Solve the following linear program by the simplex method:

$$\begin{array}{lrcl} \min & 2x_1 + 3x_2 - 7x_3 & & \\ & x_1 + x_2 + x_3 & \geq & 2, \\ \text{subject to} & 2x_1 - 5x_2 + x_3 & \geq & -1, \\ & x_1, x_2, x_3 & \geq & 0. \end{array}$$

If the problem is infeasible, say so. If it is unbounded, find vectors $u, v \in \mathbb{R}^3$ such that $u + \lambda v$ is a direction of unboundness, for $\lambda \geq 0$.

4. (15 points) Consider the following linear program:

$$\begin{array}{ll} \min\limits_{x_1, x_2, \ldots, x_n} & x_1 + 2x_2 + 3x_3 + \ldots + nx_n \\ \text{subject to} & a_1x_1 + a_2x_2 + a_3x_3 + \ldots a_nx_n \geq 1, \\ & x_1 + x_2 + x_3 + \ldots x_n \geq 1, \\ & x_1, x_2, x_3, \ldots, x_n \geq 0. \end{array}$$

where $a_1, a_2, \ldots, a_n$ are strictly positive constants that satisfy

$$a_1 > a_2 > \cdots > a_n > 1.$$

(a) Write down the dual of this problem.

(b) Show that only the *first general constraint* in the dual could possibly be active. (The "general constraints" are all constraints other than the nonnegativity constraints on the dual variables.)

(c) By inspection, find the solution of the dual. (Hint: Use part (b).)

(d) Write down the KKT conditions for this problem.

(e) Using the KKT conditions, or by any other means, find the solution of the primal.