525 Midterm Fall 2015 Solution

1 (a), (b), (c)

		$v_1 =$	$v_2 =$	$v_3 =$	$v_4 =$	
		x_1	x_2	x_3	x_4	1
$-u_1$	$y_1 =$	3	1	0	−2	−3
$-u_2$	$y_2 =$	−1	1	2	1	−1
$-u_3$	$y_3 =$	1	3	4	0	−5

2

		$v_1 =$	$u_2 =$	$v_3 =$	$v_4 =$	1
		x_1	y_2	x_3	x_4	
$-u_1$	$y_1 =$	4	1	−2	−3	−2
$-v_2$	$x_2 =$	1	1	−2	−1	1
$-u_3$	$y_3 =$	4	3	−2	−3	−2

		$u_1 =$	$u_2 =$	$v_3 =$	$v_4 =$	
		y_1	y_2	x_3	x_4	1
$-v_1$	$x_1 =$	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{2}$
$-v_2$	$x_2 =$	$\frac{1}{4}$	$\frac{3}{4}$	$-\frac{3}{2}$	$-\frac{1}{4}$	$\frac{3}{2}$
$-u_3$	$y_3 =$	1	2	0	0	0

1 (a) Row dependence: $A_{3\cdot} = A_{1\cdot} + 2A_{2\cdot}$.

(b) Column dependence revealed by v_i relationship.

1 One such relationship is

$$A_{\cdot 3} = -\tfrac{1}{2} A_{\cdot 1} + \tfrac{3}{2} A_{\cdot 2}$$

Another is

$$A_{\cdot 4} = -\tfrac{3}{4} A_{\cdot 1} + \tfrac{1}{4} A_{\cdot 2}$$

(c) Get solutions by setting $y_1 = y_2 = y_3 = 0$:

2

$$x_1 = \tfrac{1}{2} x_3 + \tfrac{3}{4} x_4 + \tfrac{1}{2}$$

$$x_2 = -\tfrac{3}{2} x_3 - \tfrac{1}{4} x_4 + \tfrac{3}{2}$$

Define $x_3 = \lambda$, $x_4 = \mu$, get

$$S = \left\{ \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1/2 \\ -3/2 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 3/4 \\ -1/4 \\ 0 \\ 1 \end{pmatrix} \text{ for } \lambda, \mu \in \mathbb{R} \right\}$$

(2)

$$\begin{array}{c|ccc|c} & x_1 & x_2 & x_3 & 1 \\ \hline x_4 = & 1 & 1 & -1 & 1 \\ x_5 = & 2 & 1 & 6 & -6 \\ \hline z = & 3 & 2 & 9 & 0 \end{array}$$

(a)

$$\begin{array}{c|ccc|c} & x_5 & x_2 & x_3 & 1 \\ \hline x_4 = & \frac{1}{2} & \frac{1}{2} & -4 & 4 \\ x_1 = & \frac{1}{2} & \frac{-1}{2} & -3 & 3 \\ \hline z = & \frac{3}{2} & \frac{1}{2} & 0 & 9 \end{array}$$

(2) optimal!

solution is $x = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$

(b) Get additional solutions by pivoting on x_3 column

$$\begin{array}{c|ccc|c} & x_5 & x_2 & x_1 & 1 \\ \hline x_4 = & \frac{-1}{6} & \frac{7}{6} & \frac{4}{3} & 0 \\ x_3 = & \frac{1}{6} & \frac{-1}{6} & -\frac{1}{3} & 1 \\ \hline z = & \frac{3}{2} & \frac{1}{2} & 0 & 9 \end{array}$$

solution: $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(2)

full set of solutions is the convex hull of these two vertices.

(2) $$\left\{ \begin{bmatrix} 3\lambda \\ 0 \\ (1-\lambda) \end{bmatrix} : \lambda \in [0,1] \right\}$$

(3)

(a)

	x_1	x_2	x_3	1
$x_4 =$	2	-3	1	-4
$x_5 =$	3	-5	1	-9
$z =$	4	6	2	0

Scheme II : Pivot x_4 with x_3

x_4 ↑ equality, x_3 ↑ free

	x_1	x_2	x_4	1
$x_3 =$	-2	3	1	4
$x_5 =$	1	-2	1	-5
$z =$	0	12	2	8

remove x_4 column

	x_1	x_2	1
$x_3 =$	-2	3	4
$x_5 =$	1	-2	-5
$z =$	0	12	8

move x_3 to end

	x_1	x_2	1
$x_5 =$	1	-2	-5
$z =$	0	12	8
$x_3 =$	-2	3	4

one step of dual simplex selects pivot (1,1)

	x_5	x_2	1
$x_1 =$	1	2	5
$z =$	0	12	8
$x_3 =$	-2	-1	-6

optimal! x :

(4)

set $x_5 = \alpha \geq 0$ to get a ray of solutions

$x_1 = \alpha + 5$

$x_2 = 0$

$x_3 = -2\alpha - 6$

$\therefore x = \left\{ \begin{pmatrix} 5 \\ 0 \\ -6 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} : \alpha \geq 0 \right\}$

(b) $\max\ 4u_1 + 9u_2$ s.t. $2u_1 + 3u_2 \le 4$

$$-3u_1 - 5u_2 \le 6$$

$$u_1 + u_2 = 2$$

2 $$u_2 \ge 0$$

(4) (a) $\max_{u \in \mathbb{R}} u$ s.t. $ua \le c$

$-uf \le -d$

$u \ge 0$

which is equivalent to

$\max_{u} u$ s.t. $ua \le c$

$uf \ge d$

(2) $u \ge 0$.

(b) No. u is bounded above by

(1) $\min_{i=1..n} \left(\frac{c_i}{a_i}\right)$

(c) Yes. If the lower bound

$\max \frac{d_i}{f_i}$

(1) is greater than upper bound $\min \frac{c_i}{a_i}$

dual can be infeasible

(d) Solution is $u^* = \min \frac{c_i}{a_i}$

provided that $\min \frac{c_i}{a_i} \ge \max \frac{d_i}{f_i}$

(1) Yes, uniquely defined by this formula.

(e) KKT conditions are

$$0 \le c - au \perp x \ge 0$$

(2) $$0 \le uf - d \perp y \ge 0$$

$$0 \le a^T x - f^T y - 1 \perp u \ge 0 .$$

need $x_i = 0$ if $\frac{c_i}{a_i} > u^*$

need $y_i = 0$ if $\frac{d_i}{f_i} < u^*$