Midterm Examination

CS 525 - Fall 2015

Monday, Nov 2, 2015. 7:15-9:15pm.

No electronic devices, notes, or books allowed, except that you may bring one standard-size sheet of paper, handwritten on both sides, into the test. **You need to give reasoning and justify all your answers**, citing the appropriate theorems where necessary.

1. Consider the following matrix and vector:

$$A = \begin{bmatrix} 3 & 1 & 0 & -2 \\ -1 & 1 & 2 & 1 \\ 1 & 3 & 4 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}.$$

   (a) Are the *rows* of the matrix A linearly independent or linearly dependent? If linearly dependent, state the linear dependence relationship between the rows.

   (b) Are the *columns* of the matrix A linearly independent or linearly dependent? If linearly dependent, state the linear dependence relationship between the columns.

   (c) Find the complete set of solutions to the linear system of equations $Ax = b$.

   (Hint: You can do (a) and (b) together by using a dual-labelled tableau.)

2. Consider the following linear program:

$$\begin{array}{lrcl} \min & 3x_1 + 2x_2 + 9x_3 & & \\ & x_1 + x_2 - x_3 & \geq & -1, \\ \text{subject to} & 2x_1 + x_2 + 6x_3 & \geq & 6, \\ & x_1, x_2, x_3 & \geq & 0. \end{array}$$

(a) Find a solution to this problem using the dual simplex method.

(b) By performing an extra pivot, find another vertex solution, and describe the full set of solutions.

3. Consider the following linear program:

$$\begin{array}{lrcl} \min & 4x_1 + 6x_2 + 2x_3 & & \\ & 2x_1 - 3x_2 + x_3 & = & 4, \\ \text{subject to} & 3x_1 - 5x_2 + x_3 & \geq & 9, \\ & x_1, x_2 & \geq & 0, \quad x_3 \text{ free.} \end{array}$$

(a) Solve this problem. If it infeasible, say so. If it is unbounded, give a direction of unboundedness. If there are multiple solutions, describe the full set of solutions. (Hint: Scheme II and the dual simplex method may be useful.)

(b) Write down the dual of this problem.

4. Consider the following linear program:

$$\min_{x,y} c^T x - d^T y \text{ subject to } a^T x - f^T y \geq 1, \quad x \geq 0, \ y \geq 0,$$

where the variables x and y are vectors with n components each (where n is some integer at least 1) and c, d, a, and f are vectors with n components whose entries are all strictly positive numbers.

(a) Write down the dual of this problem.

(b) Is it possible that for some choices of c, d, a, and f, the dual is an unbounded linear program? Explain.

(c) Are there some choices of c, d, a, and f that make the dual an *infeasible* linear program? Explain.

(d) If the strictly positive vectors c, d, a, and f are such that the dual has a solution, write it down. Is the dual solution unique?

(e) Given that the dual has a solution (as in part (d)), use the KKT conditions to say which components of the primal variable vectors x and y *must be zero* at a solution of the primal.