Chapter 3

The Simplex Method

All linear programs can be reduced to the following *standard form*

$$\begin{array}{ll} \min_x & z = p'x \\ \text{subject to} & Ax \geq b, \quad x \geq 0, \end{array} \tag{3.1}$$

where $p \in \mathbf{R}^n$, $b \in \mathbf{R}^m$ and $A \in \mathbf{R}^{m \times n}$. To create the initial tableau for the simplex method, we rewrite the problem in the following *canonical form*:

$$\begin{array}{ll} \min_{x_{\mathrm{B}}, x_{\mathrm{N}}} & z = p'x_{\mathrm{N}} + 0'x_{\mathrm{B}} \\ \text{subject to} & x_{\mathrm{B}} = Ax_{\mathrm{N}} - b, \quad x_{\mathrm{B}}, x_{\mathrm{N}} \geq 0, \end{array} \tag{3.2}$$

where the index sets N and B are defined initially as $\mathrm{N} = \{1, 2, \ldots, n\}$ and $\mathrm{B} = \{n + 1, \ldots, n + m\}$. The variables $x_{n+1}, \ldots, x_{n+m}$ are introduced to represent the slack in the inequalities $Ax \geq b$ (the difference between left- and right-hand sides of these inequalities) and are called *slack variables*. We shall represent this canonical linear program by the following tableau:

$$\begin{array}{rc|ccc|c|} & & x_1 & \cdots & x_n & 1 \\ \cline{3-6} x_{n+1} & = & A_{11} & \cdots & A_{1n} & -b_1 \\ \vdots & & \vdots & \ddots & \vdots & \vdots \\ x_{n+m} & = & A_{m1} & \cdots & A_{mn} & -b_m \\ \cline{3-6} z & = & p_1 & \cdots & p_n & 0 \\ \cline{3-6} \end{array}\ . \tag{3.3}$$

In this tableau, the slack variables $x_{n+1}, \ldots, x_{n+m}$ (the variables that make up x_{B}) are the dependent variables, while the original problem variables $x_1, \ldots, x_n$ (the variables that make up x_{N}) are independent variables. It is customary in the linear programming literature to call the dependent variables *basic* and the independent variables *nonbasic*, and we will adopt this terminology for the remainder of the book. A more succinct form of the initial tableau is known as the *condensed tableau*, which is written as follows:

$$\begin{array}{rc|c|c|} & & x_{\mathrm{N}} & 1 \\ \cline{3-4} x_{\mathrm{B}} & = & A & -b \\ \cline{3-4} z & = & p' & 0 \\ \cline{3-4} \end{array} \tag{3.4}$$

We "read" a tableau by setting the nonbasic variables x_N to zero, thus assigning the basic variables x_B and the objective variable z the values in the last column of the tableau. The tableau above represents the point $x_N = 0$ and $x_B = -b$ (that is, $x_{n+i} = -b_i$ for $i = 1, 2, \ldots, m$), with an objective of $z = 0$. The tableau is said to be *feasible* if the values assigned to the basic variables by this procedure are nonnegative. In the above, the tableau will be feasible if $b \le 0$.

At each iteration of the simplex method, we exchange one element between B and N, performing the corresponding Jordan exchange on the tableau representation, much as we did in Chapter 2 in solving systems of linear equations. We ensure that the tableau remains feasible at every iteration, and we try to choose the exchanged elements so that the objective function z decreases at every iteration. We continue in this fashion until either

1. a solution is found, or
2. we discover that the objective function is unbounded below on the feasible region, or
3. we determine that the feasible region is empty.

The simplex method can be examined from two viewpoints, which must be understood separately and jointly in order to fully comprehend the method:

1. An algebraic viewpoint represented by tableaus;
2. A geometric viewpoint obtained by plotting the constraints and the contours of the objective function in the space of original variables $\mathbf{R}^n$.

Later, we show that the points represented by each feasible tableau correspond to *vertices* of the feasible region.

3.1 A Simple Example

We now illustrate how the simplex method moves from a feasible tableau to an optimal tableau, one pivot at a time, by means of the following two-dimensional example.

Example 3.1.1.

$$\begin{array}{lrcrcr}
\min_{x_1,x_2} & 3x_1 - 6x_2 & & & & \\
\text{subject to} & x_1 & + & 2x_2 & \ge & -1 \\
 & 2x_1 & + & x_2 & \ge & 0 \\
 & x_1 & - & x_2 & \ge & -1 \\
 & x_1 & - & 4x_2 & \ge & -13 \\
 & -4x_1 & + & x_2 & \ge & -23 \\
 & & & x_1, x_2 & \ge & 0. \quad \blacksquare
\end{array}$$

The first step is to add slack variables, to convert the constraints into a set of general equalities combined with nonnegativity requirements on all the variables. The slacks are defined as follows:

$$\begin{array}{rcrcrcr}
x_3 & = & x_1 & + & 2x_2 & + & 1 \\
x_4 & = & 2x_1 & + & x_2 & & \\
x_5 & = & x_1 & - & x_2 & + & 1 \\
x_6 & = & x_1 & - & 4x_2 & + & 13 \\
x_7 & = & -4x_1 & + & x_2 & + & 23.
\end{array}$$

(When we use MATLAB to form the initial tableau, it adds the slacks automatically; there is no need to define them explicitly as above.) We formulate the initial tableau by assembling the data for the problem (that is, the matrix A and the vectors p and b) as indicated in the condensed tableau (3.4). The MATLAB command `totbl` performs this task:

```
≫ load ex3.1.1
≫ T = totbl(A,b,p);
```

$$\begin{array}{rc|cc|c|}
 & & x_1 & x_2 & 1 \\ \cline{3-5}
x_3 & = & 1 & 2 & 1 \\
x_4 & = & 2 & 1 & 0 \\
x_5 & = & 1 & -1 & 1 \\
x_6 & = & 1 & -4 & 13 \\
x_7 & = & -4 & 1 & 23 \\ \cline{3-5}
z & = & 3 & -6 & 0 \\ \cline{3-5}
\end{array}$$

The labels associated with the original and slack variables are stored in the MATLAB structure `T`. The point represented by the tableau above can be deduced by setting the nonbasic variables x_1 and x_2 both to zero. The resulting point is feasible, since the corresponding values of the basic variables, which initially are the same as the slack variables $x_3, x_4, \ldots, x_7$, are all nonnegative. The value of the objective in this tableau, $z = 0$, is obtained from the bottom right element.

We now seek a pivot—a Jordan exchange of a basic variable with a nonbasic variable—that yields a decrease in the objective z. The first issue is to choose the nonbasic variable which is to become basic, that is, to choose a pivot column in the tableau. In allowing a nonbasic variable to become basic, we are allowing its value to possibly increase from 0 to some positive value. What affect will this increase have on z and on the dependent (basic) variables? In the given example, let us try increasing x_1 from 0. We assign x_1 the (nonnegative) value λ while holding the other nonbasic variable x_2 at zero; that is,

$$x_1 = \lambda, \qquad x_2 = 0.$$

The tableau tells us how the objective z depends on x_1 and x_2, so for values given above we have

$$z = 3(\lambda) - 6(0) = 3\lambda > 0, \quad \text{for } \lambda > 0.$$

This expression tells us that z *increases* as λ increases—the opposite of what we want. Let us try instead choosing x_2 as the variable to increase, and set

$$x_1 = 0, \qquad x_2 = \lambda > 0. \tag{3.5}$$

For this choice, we have

$$z = 3(0) - 6\lambda = -6\lambda < 0 \quad \text{for } \lambda > 0,$$

thus decreasing z, as we wished. The general rule is to choose the pivot column to have a *negative* value in the last row, as this indicates that z will decrease as the variable corresponding to that column increases away from 0. We use the term *pricing* to indicate selection of the pivot column. We call the label of the pivot column the *entering variable*, as this variable is the one that "enters" the basis at this step of the simplex method.

To determine which of the basic variables is to change places with the entering variable, we examine the effect of increasing the entering variable on each of the basic variables. Given (3.5), we have the following relationships:

$$\begin{array}{rcrcr} x_3 & = & 2\lambda & + & 1 \\ x_4 & = & \lambda & & \\ x_5 & = & -\lambda & + & 1 \\ x_6 & = & -4\lambda & + & 13 \\ x_7 & = & \lambda & + & 23. \end{array}$$

Since $z = -6\lambda$, we clearly would like to make λ as large as possible, to obtain the largest possible decrease in z. On the other hand, we cannot allow λ to become *too* large, as this would force some of the basic variables to become negative. By enforcing the nonnegativity restrictions on the variables above, we obtain the following restrictions on the value of λ:

$$\begin{array}{rcrcrcrcl} x_3 & = & 2\lambda & + & 1 & \geq & 0 & \Longrightarrow & \lambda \geq -1/2 \\ x_4 & = & \lambda & & & \geq & 0 & \Longrightarrow & \lambda \geq 0 \\ x_5 & = & -\lambda & + & 1 & \geq & 0 & \Longrightarrow & \lambda \leq 1 \\ x_6 & = & -4\lambda & + & 13 & \geq & 0 & \Longrightarrow & \lambda \leq 13/4 \\ x_7 & = & \lambda & + & 23 & \geq & 0 & \Longrightarrow & \lambda \geq -23. \end{array}$$

We see that the largest nonnegative value that λ can take without violating any of these constraints is $\lambda = 1$. Moreover, we observe that the *blocking variable*—the one that will become negative if we increase λ above its limit of 1—is x_5. We choose the row for which x_5 is the label as the pivot row, and refer to x_5 as the *leaving variable*—the one that changes from being basic to being nonbasic. The pivot row selection process just outlined is called the *ratio test*.

By setting $\lambda = 1$, we have that x_1 and x_5 are zero, while the other variables remain nonnegative. We obtain the tableau corresponding to this point by performing the Jordan exchange of the row labeled x_5 (row 3) with the column labeled x_2 (column 2). The new tableau is as follows:

```
≫ T = ljx(T,3,2);
```

		x_1	x_5	1
x_3	=	3	−2	3
x_4	=	3	−1	1
x_2	=	1	−1	1
x_6	=	−3	4	9
x_7	=	−3	−1	24
z	=	−3	6	−6

Note that z has decreased from 0 to -6.

Before proceeding with this example, let us review the procedure above for a single step of the simplex method, indicating the general rules for selecting pivot columns and rows. Given the tableau

$$\begin{array}{rc|c|c|} & & x_N & 1 \\ \hline x_B & = & H & h \\ \hline z & = & c' & \alpha \\ \hline \end{array} \tag{3.6}$$

where B represents the current set of basic variables and N represents the current set of nonbasic variables, a pivot step of the simplex method is a Jordan exchange between a basic and nonbasic variable according to the following pivot selection rules:

1. *Pricing* (Selection of Pivot Column s): The pivot column is a column s with a negative element in the bottom row. These elements are called *reduced costs*.
2. *Ratio Test* (Selection of Pivot Row r): The pivot row is a row r such that

$$-h_r/H_{rs} = \min_i \{-h_i/H_{is} \mid H_{is} < 0\}$$

Note that there is considerable flexibility in selection of the pivot column, as it is often the case that many of the reduced costs are negative. One simple rule is to choose the column with the most negative reduced cost. This gives the biggest decrease in z *per unit* increase in the entering variable. However, since we cannot tell *how much* we can increase the entering variable until we perform the ratio test, it is not generally true that this choice leads to the best decrease in z on this step, among all possible pivot columns.

Returning to the example, we see that column 1, the one labeled x_1, is the only possible choice for pivot column. The ratio test indicates that row 4, labeled by x_6, should be the pivot row. We thus obtain

```
≫ T = ljx(T,4,1);
```

		x_6	x_5	1
x_3	=	-1	2	12
x_4	=	-1	3	10
x_2	=	-0.33	0.33	4
x_1	=	-0.33	1.33	3
x_7	=	1	-5	15
z	=	1	2	-15

In this tableau, all reduced costs are positive, so the pivot column selection procedure does not identify an appropriate column. This is as it should be, because this tableau is optimal! For any other feasible point than the one indicated by this tableau, we would have $x_6 \geq 0$ and $x_5 \geq 0$, giving an objective $z = x_6 + 2x_5 - 15 \geq -15$. Hence, we cannot improve z over its current value of -15 by allowing either x_5 or x_6 to enter the basis, so the tableau is optimal. The values of the basic variables can be read from the last column of the optimal tableau. We are particularly interested in the values of the two variables x_1 and x_2 from the original standard formulation of the problem; they are $x_1 = 3$ and $x_2 = 4$. In general, we have an *optimal tableau* when both the last column and the bottom row are nonnegative. (Note: when talking about the last row or last column, we do not include in our considerations the bottom right element of the tableau, the one indicating the current value of the objective. Its sign is irrelevant to the optimization process.)

Figure 3.1 illustrates Example 3.1.1.

The point labeled "Vertex 1" corresponds to the initial tableau, while "Vertex 2" is represented by the second tableau and "Vertex 3" is represented by the final tableau.

Exercise 3.1.2. Consider the problem

$$\begin{array}{ll} \min & z = p'x \\ \text{subject to} & Ax \geq b, \quad x \geq 0, \end{array}$$

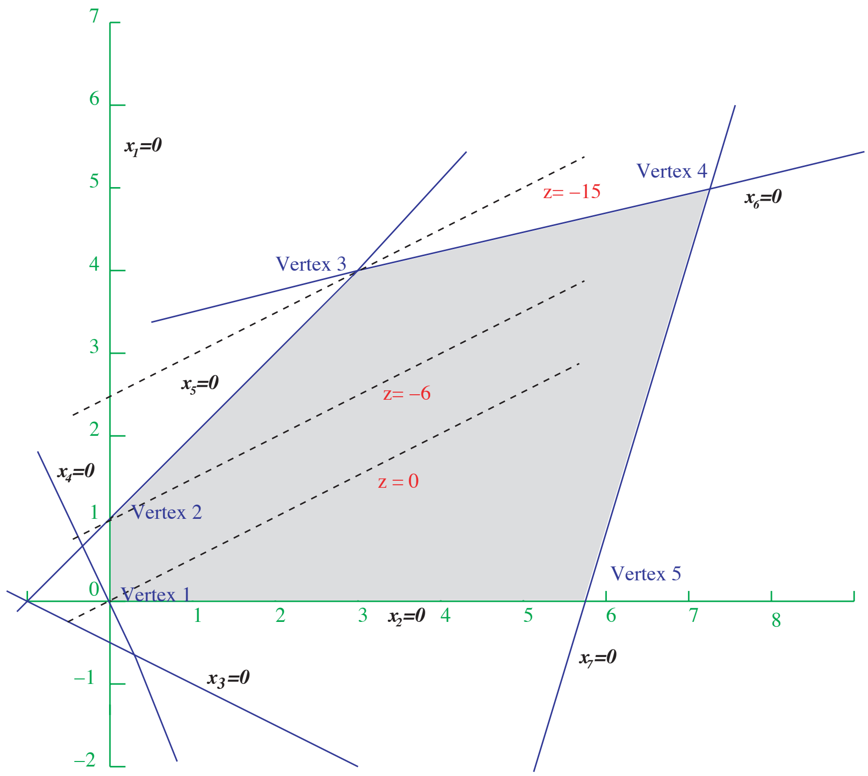

Figure 3.1. *Simplex method applied to Example 3.1.1*

where

$$A = \begin{bmatrix} 0 & -1 \\ -1 & -1 \\ -1 & 2 \\ 1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} -5 \\ -9 \\ 0 \\ -3 \end{bmatrix}, \quad p = \begin{bmatrix} -1 \\ -2 \end{bmatrix}.$$

(i) Draw the feasible region in $\mathbf{R}^2$.

(ii) Draw the contours of $z = -12$, $z = -14$, $z = -16$ and determine the solution graphically.

(iii) Solve the problem in MATLAB using Example 3.1.1 as a template. In addition, trace the path in contrasting color that the simplex method takes on your figure.