is clear that this is a feasible point for the dual problem, and also that it is a solution point. (It achieves the objective value -2, and for any other dual feasible point, we have from the objective and the first constraint that $w = -u_1 - 2u_2 \le -2$.) Hence, we conclude that the tableau is dual optimal, with solution $u_1 = 2$, $u_2 = 0$, and $w = -2$. In solving the primal problem, we have also found a solution to the dual problem.

Exercise 4-2-2. Solve the following linear program using the primal simplex method.

$$\begin{array}{lrcl} \min & z = 2x_1 + 9x_2 + 3x_3 & & \\ \text{subject to} & -x_1 - 6x_2 & \ge & -3 \\ & x_1 + 4x_2 + x_3 & \ge & 1 \\ & -2x_1 - 14x_2 & \ge & -5 \\ & x_i & \ge & 0, \quad i = 1, 2, 3. \end{array}$$

Formulate the dual of this problem and read off an optimal solution of the dual problem from the final tableau.

We now show how a primal linear program and its dual are intimately related by a number of theoretical and computational results.

4.3 Interpretation of Linear Programming Duality

We present a small example to illustrate the relationship between primal and dual linear programs, showing that these two problems arise from two different persectives on the same application. The example is a simple instance of the diet problem described in Section 1.3.1, and is a modification of the example in Winston & Venkataramanan (2003, Section 6.6).

A student is deciding what to purchase from a bakery for a tasty afternoon snack. There are two choices of food: brownies, which cost 50 cents each, and and mini-cheesecakes, which cost 80 cents. The bakery is service-oriented and is happy to let the student purchase a fraction of an item if she wishes. The bakery requires three ounces of chocolate to make each brownie (no chocolate is needed in the cheesecakes). Two ounces of sugar are needed for each brownie and four ounces of sugar for each cheesecake. Finally, two ounces of cream cheese are needed for each brownie and five ounces for each cheesecake. Being health-conscious, the student has decided that she needs at least six total ounces of sugar in her snack, along with ten ounces of sugar and eight ounces of cream cheese. She wishes to optimize her purchase by finding the least expensive combination of brownies and cheesecakes that meet these requirements. The data is summarized in Table 4.1.

	Chocolate	Sugar	Cream Cheese	Cost
Brownie	3	2	2	50
Cheesecake	0	4	5	80
Requirements	6	10	8	

Table 4.1: Data for the "Snack" Problem

The student's problem can be formulated as the classic diet problem, as follows:

$$\begin{array}{lrcl} \min_{x_1,x_2} & 50x_1 + 80x_2 & & \\ & 3x_1 & \geq & 6, \\ \text{subject to} & 2x_1 + 4x_2 & \geq & 10, \\ & 2x_1 + 5x_2 & \geq & 8, \\ & x_1, x_2 & \geq & 0, \end{array} \tag{4.4}$$

where x_1 and x_2 represent the number of brownies and cheesecakes purchased, respectively. By applying the simplex method of the previous selection, we find that the unique solution is $x = (3/2, 2)'$.

We now adopt the perspective of the wholesaler who supplies the baker with the chocolate, sugar, and cream cheese needed to make the goodies. The baker informs the supplier that he intends to purchase at least six ounces of chocolate, ten ounces of sugar, and eight ounces of cream cheese, to meet the students minimum nutritional requirements. He also shows the supplier the other data in Table 4.1. The supplier now solves the following optimization problem: How can I set the prices per ounce of chocolate, sugar, and cream cheese so that the baker will buy from me, and so that I will maximize my revenue? The baker will buy only if the total cost of raw materials for brownies is below 50 cents; otherwise he runs the risk of making a loss if the student opts to buy brownies. This restriction imposes the following constraint on the prices:

$$3u_1 + 2u_2 + 2u_3 \leq 50.$$

Similarly, he requires the cost of the raw materials for each cheesecake to be below 80 cents, leading to a second constraint:

$$4u_2 + 5u_3 \leq 80.$$

Clearly, all the prices must be nonnegative. Moreover, the revenue from the guaranteed sales is $6u_1 + 10u_2 + 8u_3$. In summary, the problem that the supplier solves to maximize his guaranteed revenue from the student's snack is as follows:

$$
\begin{array}{lrcl}
\max_{u_1,u_2,u_3} & 6u_1 + 10u_2 + 8u_3 & & \\
& 3u_1 + 2u_2 + 2u_3 & \leq & 50, \\
\text{subject to} & 4u_2 + 5u_3 & \leq & 80, \\
& u_1, u_2, u_3 & \geq & 0.
\end{array}
\tag{4.5}
$$

The solution of this problem is $u = (10/3, 20, 0)'$.

It may seem strange that the supplier charges nothing for the cream cheese ($u_3 = 0$), especially since, once he has announced his prices, the baker actually takes delivery of 11.5 ounces of it, rather than the required minimum of 8 ounces. A close examination of the problem (4.5) shows, however, that his decision is a reasonable one. If he had decided to charge a positive amount for the cream cheese (that is, $u_3 > 0$), he would have to charge less for the sugar (u_2) and possibly also for the chocolate (u_1) in order to meet the pricing constraints, and his total revenue would have been lower. Better to supply the cream cheese for free, and charge as much as possible for the sugar!

We will return to this example in the next two sections, as we develop the key results concerning the relationship between primal and dual linear programs.

4.4 Duality Theory

We begin with an elementary theorem that bounds the objectives of the dual pair of linear programs.

Theorem 4.4.1 (Weak Duality Theorem). *If x is primal feasible and u is dual feasible, then the dual objective function evaluated at u is less than or equal to the primal objective function evaluated at x, that is*

$$
\left. \begin{array}{ll} Ax \geq b, & x \geq 0 \\ A'u \leq p, & u \geq 0 \end{array} \right\rangle \implies b'u \leq p'x
$$

Proof. Note first that for any two vectors s and t of the same size, for which $s \geq 0$ and $t \geq 0$, we have $s't \geq 0$. By applying this observation to the feasibility relationships above, we have

$$
p'x = x'p \geq x'A'u = u'Ax \geq u'b = b'u
$$

where the first inequality follows from $p - A'u \geq 0$ and $x \geq 0$, and the second inequality follows from $Ax - b \geq 0$, $u \geq 0$. $\square$