CS525: ADDITIONAL NOTES ON SOLVING QPS IN NONSTANDARD FORM VIA LEMKE'S METHOD

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Abstract. We give some additional information on using Lemke's method to solve convex quadratic programs with equality constraints and free variables.

1. Introduction. Convex quadratic programs in nonstandard form give rise to LCPs in which some components of z and w are constrained to be zero, and others (their complements) are free. After setting up the tableau, "Scheme II" moves can be applied to the tableau to prepare for the application of Lemke's method, in much the same style as was done earlier for linear programming. After Scheme II is complete, we need to remember one more thing: When adding the slack variable z_0 in preparation for Phase I of Lemke, we create a new column in the tableau that contains 1 in the positions that correspond to nonnegative variables, and 0 in the positions that correspond to free variables.

We illustrate some of these concepts via an example.

$$\begin{aligned} &\min_{x_1,x_2} x_1^2 + 1.5x_1x_2 + 4x_2^2 - 3x_1 + x_2 \\ \text{subject to} \quad & 2x_1 + 3x_2 = 7, \\ & x_1 + x_2 \geq -3. \end{aligned}$$

Note that x_1 and x_2 are both free variables. The KKT conditions for this problem are as follows:

$$\begin{aligned} 2x_1 + 1.5x_2 - 2u_1 - u_2 - 3 &= 0 \\ 1.5x_1 + 8x_2 - 3u_1 - u_2 + 1 &= 0 \\ 2x_1 + 3x_2 - 7 &= 0, \\ 0 \leq x_1 + x_2 + 3 \perp u_2 &\geq 0. \end{aligned}$$

We state these conditions as an LCP in $\mathbb{R}^4$, in which $w = Mz + q$, where

$$z = \begin{bmatrix} x_1 \\ x_2 \\ u_1 \\ u_2 \end{bmatrix}, \quad M = \begin{bmatrix} 2 & 1.5 & -2 & -1 \\ 1.5 & 8 & -3 & -1 \\ 2 & 3 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \quad q = \begin{bmatrix} -3 \\ 1 \\ -7 \\ 3 \end{bmatrix},$$

where $w_1 = w_2 = w_3 = 0$ and z_1, z_2, z_3 are all free variables.

We give below an annotated Matlab diary file for setting up this problem, performing Scheme II modifications to the tableau, and applying Lemke's method.

```
% Set up tableau
>> M=[2 1.5 -2 -1; 1.5 8 -3 -1; 2 3 0 0; 1 1 0 0];
>> q=[-3 1 -7 3]';
>> T=lemketbl(M,q);
                 z1          z2          z3          z4          1
          ----------------------------------------------------------
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```
 w1 = |     2.0000     1.5000    -2.0000    -1.0000    -3.0000
 w2 = |     1.5000     8.0000    -3.0000    -1.0000     1.0000
 w3 = |     2.0000     3.0000     0.0000     0.0000    -7.0000
 w4 = |     1.0000     1.0000     0.0000     0.0000     3.0000
% Scheme II pivots! Need to get zero variablesw1, w2, w3 to the top of
% the tableau and delete them. Need to get free variables z1, z2, z3
% to the side of the tableau, and shuffle them to the bottom.
>> T=ljx(T,1,2);
             z1         w1         z3         z4          1
       -------------------------------------------------------
 z2 = |    -1.3333     0.6667     1.3333     0.6667     2.0000
 w2 = |    -9.1667     5.3333     7.6667     4.3333    17.0000
 w3 = |    -2.0000     2.0000     4.0000     2.0000    -1.0000
 w4 = |    -0.3333     0.6667     1.3333     0.6667     5.0000
>> T=ljx(T,2,1);
             w2         w1         z3         z4          1
       -------------------------------------------------------
 z2 = |     0.1455    -0.1091     0.2182     0.0364    -0.4727
 z1 = |    -0.1091     0.5818     0.8364     0.4727     1.8545
 w3 = |     0.2182     0.8364     2.3273     1.0545    -4.7091
 w4 = |     0.0364     0.4727     1.0545     0.5091     4.3818
>> T=ljx(T,3,3);
             w2         w1         w3         z4          1
       -------------------------------------------------------
 z2 = |     0.1250    -0.1875     0.0938    -0.0625    -0.0312
 z1 = |    -0.1875     0.2812     0.3594     0.0937     3.5469
 z3 = |    -0.0938    -0.3594     0.4297    -0.4531     2.0234
 w4 = |    -0.0625     0.0938     0.4531     0.0312     6.5156
% Now delete the columns for w1, w2, w3
>> T=delcol(T,'w2');
             w1         w3         z4          1
       --------------------------------------------
 z2 = |    -0.1875     0.0938    -0.0625    -0.0312
 z1 = |     0.2812     0.3594     0.0937     3.5469
 z3 = |    -0.3594     0.4297    -0.4531     2.0234
 w4 = |     0.0938     0.4531     0.0312     6.5156
>> T=delcol(T,'w1');
             w3         z4          1
       ---------------------------------
 z2 = |     0.0938    -0.0625    -0.0312
 z1 = |     0.3594     0.0937     3.5469
 z3 = |     0.4297    -0.4531     2.0234
 w4 = |     0.4531     0.0312     6.5156
>> T=delcol(T,'w3');
             z4          1
       ----------------------
 z2 = |    -0.0625    -0.0312
 z1 = |     0.0937     3.5469
 z3 = |    -0.4531     2.0234
```

```
 w4 = |     0.0312     6.5156
% rearrange rows so that the free variables appear at the bottom.
>> T=permrows(T,[4 1 2 3]);
             z4         1
       ----------------------
 w4 = |     0.0312     6.5156
 z2 = |    -0.0625    -0.0312
 z1 = |     0.0937     3.5469
 z3 = |    -0.4531     2.0234
% At this point we are done. there is no need to apply Lemke's method.
% We obtain a solution by setting
% z4=u2=0; z1=x1=3.5469; z2=x2=-.0312; z3=u1=2.0234
```

Note that this process identified the following solution to the QP

$$x = \begin{bmatrix} 3.5469 \\ -0.0312 \end{bmatrix},$$

with Lagrange multipliers (dual variables)

$$u = \begin{bmatrix} 2.0234 \\ 0 \end{bmatrix}.$$

It is easy to check that they satisfy the KKT conditions.