CS726, Fall 2007
Final Examination
Tuesday, December 21, 2007, 5:05pm–7:05pm.

Answer all FOUR questions below. One handwritten sheet of notes (written front and back) is allowed. EXPLAIN ALL YOUR ANSWERS.

1. Define the function f to be the following strictly convex quadratic:

$$f(x) = \frac{1}{2}x^T Ax + b^T x + c,$$

where $x \in \mathbf{R}^n$ and A is an $n \times n$ positive definite matrix.

   (a) Find an explicit formula for the exact minimizing α of the function

$$t(\alpha) \stackrel{\text{def}}{=} f(x + \alpha p),$$

   where x and p are vectors such that $p \neq 0$ and x is not a minimizer of f.

   (b) For what values of c_1 is the first Wolfe condition satisfied by the minimizing α from part (a)? (The first Wolfe condition is that $f(x + \alpha p) \leq f(x) + c_1 \alpha \nabla f(x)^T p$.)

2. Let $\{x_k\}$ be a sequence of vectors in $\mathbf{R}^n$ and let f be a twice continuously differentiable function.

   (a) If $\{\nabla f(x_k)\}$ has an accumulation point at 0, does it follow that the sequence $\{x_k\}$ must have a stationary accumulation point?

   (b) Suppose that $\lim_{k\to\infty} x_k = x^*$ for some x^*, that $\lim_{k\to\infty} \nabla f(x_k) = 0$, and that there is a constant $\beta > 0$ such that matrices $\nabla^2 f(x_k)$ are positive definite with

$$\|\nabla^2 f(x_k)\| \|(\nabla^2 f(x_k))^{-1}\| \leq \beta, \quad \text{for all } k > 0.$$

   Are the second-order sufficient conditions for x^* to be a local minimizer of f satisfied at x^*?

3. (a) The BFGS quasi-Newton updating formula for the approximate inverse Hessian H_k can be written as follows:

$$H_{k+1} = (I - \rho_k s_k y_k^T) H_k (I - \rho_k y_k s_k^T) + \rho_k s_k s_k^T,$$

   where

$$\rho_k = \frac{1}{y_k^T s_k}.$$

   Show that if H_k is positive definite and the curvature condition $y_k^T s_k > 0$ holds, then H_{k+1} is also positive definite.

   (b) If $y_k^T s_k \leq 0$, is it still possible for H_{k+1} to be positive definite?

4. (a) Consider the function $r : \mathbf{R} \to \mathbf{R}$ defined by $r(x) = x^q$, where q is an integer greater than 2. (Note that $x^* = 0$ is the sole root of this function and that it is degenerate, that is, $r'(x^*)$ is singular.) Show that Newton's method converges Q-linearly, and find the value of the convergence ratio (the limiting bound on $\|x_{k+1} - x^*\|/\|x_k - x^*\|$).

(b) Show that Newton's method applied to the function $r(x) = -x^5 + x^3 + 4x$ starting from $x_0 = 1$ generates a sequence of iterates that alternates between $+1$ and -1.

(c) Find the roots of the function in (b), and check that they are nondegenerate.