CS726, Fall 2008
Final Examination
Monday, December 15, 2008, 12:25pm–2:25pm.

Answer all FOUR questions below. One handwritten sheet of notes (written front and back) is allowed. EXPLAIN ALL YOUR ANSWERS.

1. (a) Consider the unconstrained minimization problem min $f(x)$, where f is a smooth function. Using the motivation from Taylor's Theorem and least-squares, derive a Barzilai-Borwein formula for the line search parameter α_k in the iteration $x_{k+1} = x_k - \alpha_k \nabla f(x_k)$.
   (b) When $f(x) = (1/2)x^T Ax$, for symmetric positive definite A, express the formula from part (a) as a function of the latest step $s_k := x_k - x_{k-1}$ and the Hessian A.
   (c) Consider the steepest descent method with exact line search applied to the convex quadratic function f from part (b). The iterations have the form $x_{k+1} = x_k - \gamma_k \nabla f(x_k)$, where γ_k is chosen to minimize the function f along the direction $-\nabla f(x_k)$. Express γ_k explicitly in terms of $s_{k+1} := x_{k+1} - x_k$ and A (using the fact that $s_{k+1} = \gamma_k \nabla f(x_k)$).
   (d) Comment on the relationship between γ_k from (c) and α_k from (b).
2. Let $f : \mathbf{R}^n \to \mathbf{R}$ be a continuously differentiable function and suppose that $\{x_k\}$ is a sequence of iterates in $\mathbf{R}^n$. Suppose further that $\liminf \|\nabla f(x_k)\| = 0$ and that $\bar{x}$ and $\tilde{x}$ are the only two accumulation points of the sequence $\{x_k\}$.
   (a) Must *at least one* of $\bar{x}$ and $\tilde{x}$ be stationary points of f? Must *both* of $\bar{x}$ and $\tilde{x}$ be stationary points of f?
   (b) How does your answer to part (a) change if $\{x_k\}$ is a *bounded* sequence?
3. A fundamental first-order necessary condition for optimality of x^* in the problem $\min_{x \in \Omega} f(x)$, where Ω is closed and convex, is that

$$x^* \in \Omega \text{ and } \nabla f(x^*)^T (z - x^*) \geq 0 \text{ for all } z \in \Omega.$$

   Find the specialization of the first-order optimality conditions to the following two definitions of Ω, where $v \in \mathbf{R}^n$ is a fixed vector:
   (a) $\Omega = \{\gamma v \,|\, \gamma \in \mathbf{R}\}$.
   (b) $\Omega = \{\gamma v \,|\, \gamma \geq 0\}$.
4. Consider the direction set $\mathcal{D} = \{p_1, p_2, \ldots, p_{n+1}\}$, where all p_i are in $\mathbf{R}^n$ with

$$p_i = e_i, \quad i = 1, 2, \ldots, n, \quad p_{n+1} = -\frac{1}{n}(1, 1, \ldots, 1)^T,$$

   where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)^T$ with the 1 in position i. Find a positive value of $\delta > 0$ such that for all possible $v \in \mathbf{R}^n$, we have

$$\max_{i=1,2,\ldots,n+1} p_i^T v \geq \delta \|v\|_1.$$