726 FALL 2013 MIDTERM.

① (a) $f(x) = \frac{1}{2} x^T A x \Rightarrow \nabla f(x) = Ax$

$$y_n = \nabla f(x_n) - \nabla f(x_{n-1})$$
$$= A(x_n - x_{n-1})$$
$$= A s_n$$

so

$$\gamma_n = \frac{s_n^T s_n}{s_n^T y_n} = \frac{s_n^T s_n}{s_n^T A s_n}$$

(b) $f(x_n - \alpha \nabla f(x_n))$

$$= \frac{1}{2}(x_n - \alpha \nabla f(x_n))^T A (x_n - \alpha \nabla f(x_n))$$

take derivative w.r.t. α and set to zero to get α_n, obtain.

$$\alpha_n = \frac{x_n^T A \nabla f(x_n)}{\nabla f(x_n)^T A \nabla f(x_n)} = \frac{x_n^T A^2 x_n}{x_n^T A^3 x_n} \quad \circledast$$

(c) note that, for the iterates generated by steepest descent, we have

$$s_{n+1} = x_{n+1} - x_n = -\alpha_n A x_n$$
$$\Leftrightarrow A x_n = \frac{-1}{\alpha_n} s_{n+1}$$

By substituting into $\circledast$ we obtain

$$\alpha_n = \frac{(1/\alpha_n^2)\, s_{n+1}^T s_{n+1}}{(1/\alpha_n^2)\, s_{n+1}^T A s_{n+1}} = \frac{s_{n+1}^T s_{n+1}}{s_{n+1}^T A s_{n+1}} = \gamma_{n+1}$$

as required

② (a) **No.** $\liminf \|\nabla f(x_k)\| = 0$ guarantees only that there is a subsequence K such that

$$\lim_{k \in K} \nabla f(x_k) = 0$$

The accumulation point may be the limit of another subsequence $\bar{K}$ for which $\nabla f(x_k)\big|_{k \in \bar{K}} \not\to 0$

(b) **Yes.** Since $\lim \nabla f(x_k) = 0$, then $\lim_{k \in K} \nabla f(x_k) = 0$

for all subsequences $K \subset \{1, 2, \ldots\}$

If $\bar{x}$ is any accumulation point, there is subsequence $\bar{K}$ such that $\lim_{k \in \bar{K}} x_k = \bar{x}$.

we have $\lim_{k \in \bar{K}} \nabla f(x_k) = 0 = \nabla f(\bar{x})$

so $\bar{x}$ is stationary

(c) We have $\nabla f(x^*) = \lim_{k \to \infty} \nabla f(x_k) = 0$.

since all $\nabla^2 f(x_k)$ are pos def, the limit is at least positive semidefinite. (The minimum eigenvalue of $\nabla f(x_k)$ is positive for all k - it may approach zero as $k \to \infty$ but cannot become negative

(d) **No.** We can have $\lambda_{min}(\nabla^2 f(x_k)) \to 0$, so $\nabla^2 f(x^*)$ may be only positive semidefinite, not positive definite

(3) (a) Suppose there exists v such that $v^T H_{k+1} v = 0$.
Then

$$0 = v^T H_{k+1} v = v^T \left[I - \rho_k s_k y_k^T\right] H_k \left[I - \rho_k y_k s_k^T\right] v + \rho_k v^T s_k s_k^T v$$
$$= (v - \rho_k (s_k^T v) y_k)^T H_k (v - \rho_k (s_k^T v) y_k) + \rho_k (v^T s_k)^2$$

Since $\rho_k = \frac{1}{s_k^T y_k} > 0$, and since H_k is pos def,
we must have

$$v - \rho_k (s_k^T v) y_k = 0$$

and $s_k^T v = 0$.

It follows by substituting the second expression into the first that $v = 0$.

(b) No. By secant condition we have

$$H_{k+1} y_k = s_k.$$

Thus, by taking inner product of both sides with y_k, we obtain

$$y_k^T H_{k+1} y_k = y_k^T s_k \leq 0,$$

So H_{k+1} cannot be positive definite

(4) $r_{k+1} = Ax_{k+1} - b = Ax_k - b + A(x_{k+1} - x_k) = r_k + A(\alpha_k p_k)$

Taking product of both sides with p_k, we obtain

$$p_k^T r_{k+1} = p_k^T r_k + \alpha_k p_k^T A p_k = 0 \quad , \text{ by definition of } \alpha_k$$

Thus $r_1^T p_0 = 0$, and the claim is satisfied for $k=1$

Suppose that $r_k^T p_j = 0$, $j = 0, 1, \ldots k-1$, for some k.
To complete the inductive proof, we need

$$r_{k+1}^T p_j = 0 \quad , j = 0, 1, \ldots k$$

we have proved this already for $j = k$
For $j = 0, 1, 2, \ldots k-1$, take inner product of the expression above with p_j to obtain

$$p_j^T r_{k+1} = \underbrace{p_j^T r_k}_{\substack{=0 \\ \text{by inductive} \\ \text{hypothesis}}} + \alpha_k \underbrace{p_j^T A p_k}_{\substack{=0 \text{ by} \\ \text{conjugacy}}} = 0 .$$

as required