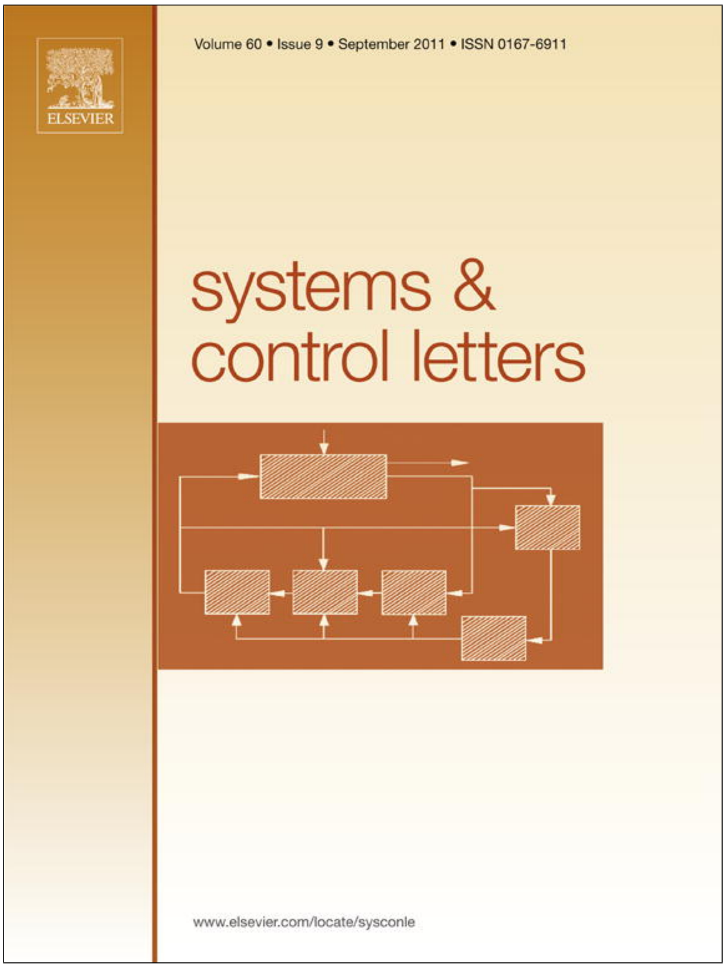
Volume 60 • Issue 9 • September 2011 • ISSN 0167-6911
ELSEVIER
systems & control letters
www.elsevier.com/locate/sysconle

Systems & Control Letters 60 (2011) 747–755

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Conditions under which suboptimal nonlinear MPC is inherently robust☆

Gabriele Pannocchia [a,*], James B. Rawlings [b], Stephen J. Wright [c]

[a] *Dip. Ing. Chim., Chim. Ind. e Sc. Mat. (DICCISM), University of Pisa, Pisa, Italy*

[b] *Department of Chemical and Biological Engineering, University of Wisconsin, Madison, WI, USA*

[c] *Computer Sciences Department, University of Wisconsin, Madison, WI, USA*

ARTICLE INFO

Article history:
Received 18 February 2011
Accepted 18 May 2011
Available online 14 July 2011

Keywords:
Robust stability
Suboptimal model predictive control
Difference inclusions
Inherent robustness
Lyapunov functions

ABSTRACT

We address the *inherent robustness* properties of nonlinear systems controlled by suboptimal model predictive control (MPC), i.e., when a *suboptimal* solution of the (generally nonconvex) optimization problem, rather than an element of *the optimal* solution set, is used for the control. The suboptimal control law is then a set-valued map, and consequently, the closed-loop system is described by a difference inclusion. Under mild assumptions on the system and cost functions, we establish nominal exponential stability of the equilibrium, and with a continuity assumption on the feasible input set, we prove robust exponential stability with respect to small, but otherwise arbitrary, additive process disturbances and state measurement/estimation errors. These results are obtained by showing that the suboptimal cost is a *continuous exponential* Lyapunov function for an appropriately augmented closed-loop system, written as a difference inclusion, and that recursive feasibility is implied by such (nominal) exponential cost decay. These novel robustness properties for suboptimal MPC are inherited also by optimal nonlinear MPC. We conclude the paper by showing that, in the absence of state constraints, we can replace the terminal constraint with an appropriate terminal cost, and the robustness properties are established on a set that approaches the nominal feasibility set for small disturbances. The somewhat surprising and satisfying conclusion of this study is that *suboptimal MPC has the same inherent robustness properties as optimal MPC.*

1. Introduction

The stability properties of model predictive control (MPC) have been investigated extensively during the past two decades, and it is generally known that the equilibria of constrained nonlinear systems in a closed loop with suitably designed MPC are asymptotically stable [1]. The usual path to prove asymptotic stability of an equilibrium point requires that the optimal MPC cost function is a Lyapunov function for the closed-loop system, and this usually implies the use of an appropriate terminal cost (with or without an associated terminal constraint). Different flavors can be considered along these lines, and the interested reader may find them in [2, Ch. 2] and references therein.

In order to guarantee asymptotic stability, most MPC formulations assume that the optimal control problem is solved *exactly*. While this assumption may be acceptable for linear systems with linear or quadratic cost functions (hence the associated optimal control problem is an LP or a QP, respectively), for nonlinear systems, there is usually no guarantee that the global optimum can be achieved, unless the resulting optimization problem is convex [3]. If, on the other hand, a *suboptimal* solution of the control problem is used in the closed loop, stability may not hold. Even if it *does* hold, it may be difficult to establish because the associated control law is no longer a function of solely the current state, and hence the cost function is not a Lyapunov function for the closed-loop system. However, as established by [4], if the optimization provides a feasible, suboptimal solution that only *improves* the cost from a well-chosen warm start (and some other more technical assumptions are satisfied), asymptotic stability of the equilibrium can be established. (See also [2, Sections 2.8 and 6.1.2] for further details on stability under suboptimal MPC.)

When considering systems perturbed by additive process disturbances and/or state measurement (or estimation) errors, in most of the literature, the perturbations have been directly taken into account in the controller formulation leading to the so-called robust MPC formulations (see e.g. [5–8], [2, Ch. 3] and references therein). Even for linear systems, robust MPC formulations are often more demanding from a computational point of view, and, more importantly, they tend to be conservative in order to preserve recursive feasibility in worst-case scenarios. A major challenge in robust MPC design is handling hard state constraints. For example, to maintain feasibility of state constraints under disturbances, [9] propose modifying the nominal MPC problem by altering the state constraints so that they become progressively tighter with time.

☆ This work was supported by National Science Foundation (Grant CTS-0456694) and TWCCC members.

* Corresponding author. Tel.: +39 050 2217838; fax: +39 050 2217866.
E-mail addresses: g.pannocchia@diccism.unipi.it (G. Pannocchia), rawlings@engr.wisc.edu (J.B. Rawlings), swright@cs.wisc.edu (S.J. Wright).

0167-6911/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.sysconle.2011.05.013

On the other hand, *inherent robustness* properties, i.e., the stability properties of a perturbed system in a closed loop with a *nominal* MPC design *ignoring* disturbances, which includes most industrial MPC applications, have received much less attention [10,11], as pointed out by Teel and coworkers [12]. In particular, [12] presented examples of nonlinear systems controlled by MPC in which the asymptotic stability of the equilibrium is destroyed by arbitrarily small perturbations. They prove that the existence of a continuous Lyapunov function for the closed-loop system implies robustness to sufficiently small perturbations, provided that the optimization problem remains feasible at all times. In a subsequent paper [13], they presented sufficient conditions for robust stability of MPC using the time-varying tightening of the state constraints to maintain feasibility. Further results on robustness of discontinuous discrete-time systems and Lyapunov functions were discussed in [14]. In [12], the authors also show that for *linear* systems with a quadratic cost, the optimal MPC cost function is a continuous Lyapunov function for the closed-loop system, because the optimal state-feedback law is continuous, hence achieving inherent robust stability of the equilibrium. This result is extended to the entire domain of the value function in [2, Proposition 7.13]. On the other hand, a suboptimal MPC law is not necessarily continuous, even for linear systems, and hence inherent robustness cannot be established even in such a *simple* case. In a significant paper, Lazar and Heemels were the first to address explicitly the robustness of suboptimal MPC [15]. Their robustness results apply to a class of suboptimal MPC algorithms, which satisfy a *specified* degree of suboptimality. They also employed the time-varying state constraint tightening approach of [9,13] to achieve recursive feasibility under disturbances. A detailed comparison between the results and approach considered here and that of [15] is given in the conclusions.

With these observations in mind, the objective of this paper is to present novel results on inherent robust stability of a class of nonlinear systems controlled by a general and implementable suboptimal MPC. To do so, we treat the suboptimal control law as a set-valued map and consequently we use difference inclusions [16] to describe the closed-loop system. Furthermore, we show that such robustness is achieved with respect to sufficiently small but arbitrary perturbations, without requiring *a priori* recursive feasibility. Our nominal MPC controller design does *not* employ any constraint tightening procedure to ensure feasibility. We instead resolve infeasibilities, if they occur, online. It should also be mentioned that, in application, all state constraint-based controllers, even the robust designs, must include a provision for resolving infeasibilities online simply because the *encountered* disturbances can never be guaranteed to lie within the *designed* disturbance sets.

As a corollary to these results, we note that optimal MPC is *a special case* of suboptimal MPC, and thus the inherent robust stability properties are consequently established for the nominal, optimal controller. This represents itself a novel contribution of this paper. Most of these results can be tailored to the case in which the controlled system is linear, and the specialization to partial enumeration MPC [17] will be the subject of a separate publication [18].

Notation. The symbols $\mathbb{I}_{\geq 0}$ and $\mathbb{R}_{\geq 0}$ denote the sets of nonnegative integers and reals, respectively. The symbol $\mathbb{I}_{0:N-1}$ denotes the set $\{0, 1, \dots, N-1\}$. The symbol $|\cdot|$ denotes the Euclidean norm and $\mathbb{B}$ denotes the closed ball of radius 1 centered around the origin. If X and Y are two subsets of the same space, we define $Z = X \oplus Y := \{z = x+y \mid x \in X, y \in Y\}$. We denote the interior of a set X as $\mathrm{int}(X)$. Given a nonnegative function $V : X \to \mathbb{R}_{\geq 0}$ and a positive scalar α, we define $\mathrm{lev}_\alpha V := \{x \in X \mid V(x) \leq \alpha\}$.

2. Basic definitions and assumptions

2.1. MPC problem and optimal solution

In this paper, we consider discrete-time systems in the form:

$$x^+ = f(x, u) \tag{1}$$

in which $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state and the input at a given time, while $x^+ \in \mathbb{R}^n$ is the successor state. Both state and input are subject to constraints:

$$x(k) \in \mathbb{X}, \qquad u(k) \in \mathbb{U} \quad \text{for all } k \in \mathbb{I}_{\geq 0}.$$

Given an integer N (referred to as the finite horizon), and an input sequence $\mathbf{u}$ of length N, $\mathbf{u} = \{u(0), u(1), \dots, u(N-1)\}$, let $\phi(k; x, \mathbf{u})$ denote the solution of (1) at time k for a given initial state $x(0) = x$. Then we define the set of feasible initial states and associated control sequences:

$$\mathbb{Z}_N := \{(x, \mathbf{u}) \mid u(k) \in \mathbb{U}, \phi(k; x, \mathbf{u}) \in \mathbb{X} \text{ for all } k \in \mathbb{I}_{0:N-1}, \text{ and } \phi(N; x, \mathbf{u}) \in \mathbb{X}_f\}$$

in which $\mathbb{X}_f \subseteq \mathbb{X}$ is the set of feasible states at the end of the finite horizon. Consequently, we can define the set of feasible initial states as:

$$\mathcal{X}_N := \{x \in \mathbb{R}^n \mid \exists \mathbf{u} \in \mathbb{U}^N \text{ such that } (x, \mathbf{u}) \in \mathbb{Z}_N\} \tag{2}$$

and for each $x \in \mathcal{X}_N$, the corresponding set of feasible input sequences is defined as:

$$\mathcal{U}_N(x) := \{\mathbf{u} \mid (x, \mathbf{u}) \in \mathbb{Z}_N\}.$$

For any state $x \in \mathbb{R}^n$ and input sequence $\mathbf{u} \in \mathbb{U}^N$, we define[1]:

$$V_N(x, \mathbf{u}) := \sum_{k=0}^{N-1} \ell(\phi(k; x, \mathbf{u}), u(k)) + V_f(\phi(N; x, \mathbf{u}))$$

and we consider the finite horizon optimal control problem:

$$\mathbb{P}_N(x) : \min_{\mathbf{u}} V_N(x, \mathbf{u}) \quad \text{s.t. } \mathbf{u} \in \mathcal{U}_N(x).$$

We make the following basic assumptions.

Assumption 1. The functions $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ and $V_f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ are continuous, $f(0, 0) = 0$, $\ell(0, 0) = 0$ and $V_f(0) = 0$.

Assumption 2. (a) The set $\mathbb{U}$ is compact and contains the origin. The sets $\mathbb{X}$ and $\mathbb{X}_f$ are closed and contain the origin in their interiors, $\mathbb{X}_f \subseteq \mathbb{X}$.
(b) The set $\mathbb{U}$ is compact and contains the origin. The sets $\mathbb{X} = \mathbb{R}^n$ and $\mathbb{X}_f = \mathrm{lev}_\alpha V_f = \{x \in \mathbb{R}^n \mid V_f(x) \leq \alpha\}$, with $\alpha > 0$.

Assumption 3. For any $x \in \mathbb{X}_f$, there exists $u \in \mathbb{U}$ such that $f(x, u) \in \mathbb{X}_f$ and $V_f(f(x, u)) + \ell(x, u) \leq V_f(x)$.

Assumption 4. There exist positive constants a, a_1', a_2', a_f and $\bar{r}$, such that the cost function satisfies the inequalities

$$\ell(x, u) \geq a_1' |(x, u)|^a \quad \text{for all } (x, u) \in \mathbb{X} \times \mathbb{U}$$
$$V_N(x, \mathbf{u}) \leq a_2' |(x, \mathbf{u})|^a \quad \text{for all } (x, \mathbf{u}) \in \bar{r}\mathbb{B}$$
$$V_f(x) \leq a_f |x|^a \quad \text{for all } x \in \mathbb{X}.$$

Remark 5. In Assumption 2, we allow the origin to be on the boundary of the (deviation) input space $\mathbb{U}$. This case often arises in industrial applications where an upper (typically economic) optimization layer pushes the desired steady-state operating point to the boundary of the (absolute) input space.

[1] Notice that we define $V_N(\cdot)$ even for pairs $(x, \mathbf{u}) \notin \mathbb{Z}_N$. This extension is necessary for robust stability analysis of the perturbed case.

Remark 6. Assumption 2(b) (with Assumption 1) implies Assumption 2(a); we use Assumption 2(b) to treat the case without state constraints in Section 5.

Remark 7. Assumption 3 implies that $\mathbb{X}_f$ is control invariant and $\mathbb{X}_f \subseteq \mathcal{X}_N$. We can define a terminal control law (set-valued map) $\kappa_f(\cdot)$ on $\mathbb{X}_f$ so that $u \in \kappa_f(x)$ satisfies Assumption 3.

Let $\mathbf{u}^0(x)$ be the *optimal* solution of $\mathbb{P}_N(x)$ and let $\kappa_N(\cdot) := \mathbf{u}^0(0; \cdot)$ denote its first component, written as an implicit function[2] of x. The closed-loop evolution of *nominal* system (1) under optimal MPC can be written as

$$x^+ = f(x, \kappa_N(x)). \tag{3}$$

2.2. Suboptimal MPC

Rather then solving $\mathbb{P}_N(x)$ *exactly*, we consider using any (unspecified) suboptimal algorithm having the following properties. Let $\mathbf{u} \in \mathcal{U}_N(x)$ denote the (suboptimal) control sequence for the initial state x, and let $\tilde{\mathbf{u}}$ denote a *warm start* for the successor initial state $x^+ = f(x, u(0; x))$, obtained from $(x, \mathbf{u})$ by

$$\tilde{\mathbf{u}} := \{u(1; x), u(2; x), \dots, u(N-1; x), u_+\} \tag{4}$$

in which $u_+ \in \mathbb{U}$ is any input that satisfies the invariance conditions of Assumption 3 for $x = \phi(N; x, \mathbf{u}) \in \mathbb{X}_f$, i.e., $u_+ \in \kappa_f(\phi(N; x, \mathbf{u}))$. We observe that the warm start satisfies $\tilde{\mathbf{u}} \in \mathcal{U}_N(x^+)$. Then, the suboptimal input sequence for any given $x^+ \in \mathcal{X}_N$ is defined as *any* $\mathbf{u}^+ \in \mathbb{U}^N$ that satisfies:

$$\mathbf{u}^+ \in \mathcal{U}_N(x^+) \tag{5a}$$

$$V_N(x^+, \mathbf{u}^+) \le V_N(x^+, \tilde{\mathbf{u}}) \tag{5b}$$

$$V_N(x^+, \mathbf{u}^+) \le V_f(x^+) \quad \text{when } x^+ \in r\mathbb{B} \tag{5c}$$

in which r is a positive scalar sufficiently small that $r\mathbb{B} \subseteq \mathbb{X}_f$. Notice that constraint (5c) is required to hold only if $x^+ \in r\mathbb{B}$, and, as proved in Lemma 16, it implies that $|\mathbf{u}^+| \to 0$ as $|x^+| \to 0$.

Remark 8. Condition (5b) ensures that the computed suboptimal cost is no larger than that of the warm start.

Proposition 9. *Any $\mathbf{u}^0(x^+)$, optimal solution to $\mathbb{P}_N(x^+)$, satisfies conditions (5a) and (5b) for all $x^+ \in \mathcal{X}_N$. Moreover, the inequality in condition (5c) is satisfied by $\mathbf{u}^0(x^+)$ for all $x^+ \in \mathbb{X}_f$.*

Proof. Satisfaction of (5a) and (5b) by $\mathbf{u}^0(x^+)$ is implied by the optimality of $\mathbf{u}^0(x^+)$. For the final claim, consider any $x^+ \in \mathbb{X}_f$, define $x(0) := x^+$ and choose any $u(0) \in \kappa_f(x(0))$ satisfying Assumption 3. We thus obtain $V_f(x(1)) + \ell(x(0), u(0)) \le V_f(x(0))$. Because $x(1) \in \mathbb{X}_f$, we can choose $u(1) \in \kappa_f(x(1))$ satisfying Assumption 3 to obtain $V_f(x(2)) + \ell(x(1), u(1)) + \ell(x(0), u(0)) \le V_f(x(1)) + \ell(x(0), u(0)) \le V_f(x(0))$. Continuing in this fashion for $k = 2, 3, \dots, N-1$, and defining $\mathbf{u}_f = (u(0), u(1), \dots, u(N-1))$, we obtain $V_N(x^+, \mathbf{u}_f) \le V_f(x^+)$. Finally, optimality of $\mathbf{u}^0(x^+)$ implies that (5c) holds for $\mathbf{u}^0(x^+)$. □

Corollary 10. *For any $x^+ \in \mathcal{X}_N$, there exists a $\mathbf{u}^+ \in \mathcal{U}_N(x^+)$ satisfying all conditions (5) for all $\tilde{\mathbf{u}} \in \mathcal{U}_N(x^+)$.*

We now observe that $\mathbf{u}^+$ is a set-valued map of the state x^+, and so is the associated first component $u(0; x^+)$. If we, again, denote the latter map as $\kappa_N(\cdot)$, we can write the evolution of the closed-loop system as the following difference inclusion:

$$x^+ \in F(x) := \{f(x, u) \mid u \in \kappa_N(x)\}. \tag{6}$$

[2] Existence of the optimal solution can be established [2, Prop. 2.4]. Uniqueness is not guaranteed in general, however, and $\kappa_N(\cdot)$ may be a set-valued map. This possibility is discussed in detail in the suboptimal MPC case covered next.

Proposition 11. *We have that $\kappa_N(0) = \{0\}$ and $F(0) = \{0\}$.*

Proof. From $\ell(x, u) \ge a_1' |(x, u)|^a$ and from $V_f(x) \ge 0$, we have

$$V_N(x, \mathbf{u}) \ge a_1' \sum_{k=0}^{N-1} |(\phi(k; x, \mathbf{u}), u(k))|^a$$
$$\ge a_1' \left[|(x, u(0))|^a + \sum_{k=1}^{N-1} |u(k)|^a \right] \ge a_1' N^{-a} |(x, \mathbf{u})|^a,$$

where the last inequality is from Lemma 43. Thus, choosing a_1 to satisfy $0 < a_1 \le N^{-a} a_1'$, we have that $V_N(x, \mathbf{u}) \ge a_1 |(x, \mathbf{u})|^a$ for all $(x, \mathbf{u}) \in \mathbb{X} \times \mathbb{U}^N$. From (5c), Assumptions 1 and 4, we have that $a_1 |(0, \mathbf{u})|^a \le V_N(0, \mathbf{u}) \le V_f(0) = 0$. Thus, it follows that $\mathbf{u} = 0$ and hence $\kappa_N(0) = u(0; 0) = \{0\}$. The second result then follows from Assumption 1. □

2.3. Exponential stability for difference inclusions

Motivated by the description as difference inclusion of the closed-loop system evolution under suboptimal MPC, we consider the following definitions. Given a difference inclusion[3] $z^+ \in H(z)$, $H(0) = \{0\}$, we denote by $\psi(k; z) := z(k)$ *a solution* at time $k \in \mathbb{I}_{\ge 0}$ starting from the initial state $z(0) = z$.

Definition 12 (*Exponential Stability*)**.** The origin of the difference inclusion $z^+ \in H(z)$ is *exponentially stable* (ES) on $\mathcal{Z}$, $0 \in \mathcal{Z}$, if there exist scalars $b > 0$ and $0 < \lambda < 1$, such that for any $z \in \mathcal{Z}$, *all* solutions $\psi(k; z)$ satisfy:

$$\psi(k; z) \in \mathcal{Z}, \quad |\psi(k; z)| \le b\lambda^k |z| \quad \text{for all } k \in \mathbb{I}_{\ge 0}.$$

Definition 13 (*Exponential Lyapunov Function*)**.** V is an *exponential Lyapunov* function on the set $\mathcal{Z}$ for the difference inclusion $z^+ \in H(z)$ if there exist positive scalars a, a_1, a_2 and a_3 such that the following holds for all $z \in \mathcal{Z}$

$$a_1 |z|^a \le V(z) \le a_2 |z|^a, \quad \max_{z^+ \in H(z)} V(z^+) \le V(z) - a_3 |z|^a.$$

We have the following results.

Proposition 14. *If V is an exponential Lyapunov function on the set $\mathcal{Z}$ for the difference inclusion $z^+ \in H(z)$, there exists $0 < \gamma < 1$ such that:*

$$\max_{z^+ \in H(z)} V(z^+) \le \gamma V(z).$$

Proof. From the definition of V, $z \in \mathcal{Z}$ implies that

$$\max_{z^+ \in H(z)} V(z^+) \le V(z) - a_3 |z|^a \le V(z) - \frac{a_3}{a_2} V(z) \le \gamma V(z)$$

for $\gamma > 1 - a_3/a_2$. Since $a_2 \ge a_3 > 0$, we have $0 < \gamma < 1$. □

Lemma 15. *If the set $\mathcal{Z}$, $0 \in \mathcal{Z}$, is positively invariant for the difference inclusion $z^+ \in H(z)$, $H(0) = \{0\}$, and there exists an exponential Lyapunov function V on $\mathcal{Z}$, the origin is ES on $\mathcal{Z}$.*

Proof. Since $\psi(k; z) \in \mathcal{Z}$ for all $k \in \mathbb{I}_{\ge 0}$, and using Proposition 14, we write: $|\psi(k; z)|^a \le \frac{V(\psi(k; z))}{a_1} \le \frac{\gamma^k V(z)}{a_1} \le \frac{\gamma^k a_2 |z|^a}{a_1}$. Thus, we obtain: $|\psi(k; z)| \le b\lambda^k |z|$ in which $\lambda = \gamma^{1/a}$ and $b = \left(\frac{a_2}{a_1}\right)^{1/a}$, and we note that $0 < \lambda < 1$. □

[3] For reasons that will be clear later on, here we use z to denote the state.

3. Nominal exponential stability of suboptimal MPC

We now present the first set of novel results for discrete-time systems in closed loop with suboptimal MPC.

3.1. Extended state and supporting results

We consider an *extended* state $z = (x, \mathbf{u})$, and we observe that it evolves according to the following difference inclusion:

$$z^+ \in H(z) := \{(x^+, \mathbf{u}^+) \mid x^+ = f(x, u(0; x)), \mathbf{u}^+ \in G(z)\} \quad (7)$$

in which (notice that both x^+ and $\tilde{\mathbf{u}}$ depend on z):

$$G(z) := \{\mathbf{u}^+ \mid \mathbf{u}^+ \in \mathcal{U}_N(x^+), V_N(x^+, \mathbf{u}^+) \le V_N(x^+, \tilde{\mathbf{u}}), \text{ and } V_N(x^+, \mathbf{u}^+) \le V_f(x^+) \text{ if } x^+ \in r\mathbb{B}\}.$$

We also define the following set (notice that $r\mathbb{B} \subseteq \mathbb{X}_f$):

$$\mathcal{Z}_r := \{(x, \mathbf{u}) \in \mathbb{Z}_N \mid V_N(x, \mathbf{u}) \le V_f(x) \text{ if } x \in r\mathbb{B}\}.$$

Lemma 16. *There exists a positive constant c such that $|\mathbf{u}| \le c|x|$ for any $(x, \mathbf{u}) \in \mathcal{Z}_r$.*

Proof. We first show that $|\mathbf{u}| \le \bar{c}|x|$ holds, for some $\bar{c}$, if $x \in r\mathbb{B} \subseteq \mathbb{X}_f$. Recall from the proof of Proposition 11 that there is $a_1 > 0$ such that $a_1|(x, \mathbf{u})|^a \le V_N(x, \mathbf{u})$ for all $(x, \mathbf{u}) \in \mathbb{X} \times \mathbb{U}^N$. For $x \in r\mathbb{B} \subseteq \mathbb{X}_f \subseteq \mathbb{X}$, we can therefore write:

$$a_1|\mathbf{u}|^a \le a_1|(x, \mathbf{u})|^a \le V_N(x, \mathbf{u}) \le V_f(x) \le a_f|x|^a.$$

Thus, given any $\bar{c} \ge (a_f/a_1)^{(1/a)}$, we obtain $|\mathbf{u}| \le \bar{c}|x|$ for any $x \in r\mathbb{B}$. Define $\mu = \max_{\mathbf{u}\in\mathbb{U}^N} |\mathbf{u}|$, and note that $\mu < \infty$ because $\mathbb{U}^N$ is compact. Choosing $c \ge \max\{\frac{\mu}{r}, \bar{c}\}$, we observe that $|\mathbf{u}| \le c|x|$ for all $(x, \mathbf{u}) \in \mathcal{Z}_r$. In fact, if $x \in r\mathbb{B}$, we have that $|\mathbf{u}| \le \bar{c}|x| \le c|x|$; while if $x \notin r\mathbb{B}$, we have that $|\mathbf{u}| \le \mu \le \frac{\mu|x|}{r} \le c|x|$. □

Lemma 17. *$V_N(\cdot)$ is an exponential Lyapunov function for extended closed-loop system (7) in any compact subset of $\mathcal{Z}_r$.*

Proof. As established in the proof of Proposition 11, we have that $a_1|z|^a \le V_N(z)$ for some $a_1 > 0$ and all $z \in \mathcal{Z}_r$. Consider any compact set $\mathcal{C} \subseteq \mathcal{Z}_r$ and define: $\mu := \max_{z\in\mathcal{C}} V_N(z)$. Note that from Assumption 1, it follows that $V_N(\cdot)$ is continuous; thus, μ is well defined. From Assumption 4, if we choose $a_2 \ge \max\{\mu/\bar{r}^a, a'_2\}$, we have that:

$$V_N(z) \le a_2|z|^a \quad \text{for all } z \in \mathcal{C}.$$

We verify this fact by noting that if $z \in \bar{r}\mathbb{B} \cap \mathcal{Z}_r$, we have from Assumption 4 that $V_N(z) \le a'_2|z|^a \le a_2|z|^a$; if instead $z \in \mathcal{Z}_r \setminus \bar{r}\mathbb{B}$, we have that $V_N(z) \le \mu \le \mu|z|^a/\bar{r}^a \le a_2|z|^a$. We now prove that $V_N(z^+) \le V_N(z) - a_3|z|^a$ for all $z^+ \in H(z)$ and $z \in \mathcal{Z}_r$. In fact, for all $z^+ \in H(z)$, we have from Assumption 4 that

$$V_N(z^+) \le V_N(z) - \ell(x, u(0)) \le V_N(z) - a'_1|(x, u(0))|^a.$$

From Lemma 16 we can write:

$$|z| \le (|x| + |\mathbf{u}|) \le (1 + c)|x| \le (1 + c)|(x, u(0))|.$$

Thus, if we define a positive constant $a_3 \le \frac{a'_1}{(1+c)^a}$, we can write:

$$V_N(z^+) \le V_N(z) - a'_1|(x, u(0))|^a \le V_N(z) - \frac{a'_1}{(1 + c)^a}|z|^a \le V_N(z) - a_3|z|^a$$

for all $z^+ \in H(z)$ and $z \in \mathcal{Z}_r$. □

3.2. Main results

We now state the first main result of this paper about nominal stability of the origin under suboptimal MPC.

Theorem 18 (*ES*). *Under Assumptions 1, 2(a), 3 and 4, the origin of the closed-loop system (6) is ES on (arbitrarily large) compact subsets of $\mathcal{X}_N$.*

Proof. From Lemma 17, we have that $V_N(\cdot)$ is an exponential Lyapunov function for (7) in any given compact subset of $\mathcal{Z}_r$. Let $\bar{V}$ be an arbitrary positive scalar, and consider the set

$$\mathcal{S} = \{(x, \mathbf{u}) \in \mathcal{Z}_r \mid V_N(x, \mathbf{u}) \le \bar{V}\}.$$

We observe that $\mathcal{S} \subseteq \mathcal{Z}_r$ is compact and is invariant for (7). By Lemma 15, these facts prove that the origin of the extended system (7) is ES on $\mathcal{S}$, i.e., there exist scalars $b' > 0$ and $0 < \lambda < 1$, such that for any $z \in \mathcal{S}$ we can write:

$$\psi(k; z) \in \mathcal{S} \quad \text{and} \quad |\psi(k; z)| \le b'\lambda^k|z| \quad \text{for all } k \in \mathbb{I}_{\ge 0}$$

in which $\psi(k; z) = z(k)$ is *a solution* of (7) at time k for a given initial extended state $z(0) = z$. We define $\mathcal{C} := \{x \in \mathcal{X}_N \mid \exists \mathbf{u} \in \mathcal{U}_N(x)$ such that $(x, \mathbf{u}) \in \mathcal{S}\}$ and we note that $\mathcal{C} \subseteq \mathcal{X}_N$ and that $\mathcal{C}$ is compact because it is the projection onto $\mathbb{R}^n$ of the compact set $\mathcal{S}$. Thus for any $x \in \mathcal{C}$ and its associated suboptimal input sequence $\mathbf{u}$ such that $z = (x, \mathbf{u}) \in \mathcal{S}$, we denote with $\phi(k; x)$ the state component of $\psi(k; z)$, i.e., *a solution* of nonextended system (6), and for all $k \in \mathbb{I}_{\ge 0}$ we can write:

$$\phi(k; x) \in \mathcal{C} \quad \text{and} \quad |\phi(k; x)| \le |\psi(k; z)| \le b'\lambda^k|z| \le b\lambda^k|x|$$

in which $b = b'(1 + c)$, because from Lemma 16 it follows that $|z| \le |x| + |\mathbf{u}| \le (1 + c)|x|$. This concludes the proof because it states that the origin of the closed-loop system (6) is ES on $\mathcal{C}$, and $\bar{V}$ can be chosen large enough for $\mathcal{C}$ to contain any given compact subset of $\mathcal{X}_N$. □

Corollary 19. *Under Assumptions 1, 2(a), 3 and 4, if $\mathcal{X}_N$ is compact, the origin of the closed-loop system (6) is ES on $\mathcal{X}_N$.*

4. Robust exponential stability of suboptimal MPC

4.1. Disturbances and robust stability definitions

For robustness analysis, we consider the closed-loop evolution of the *perturbed* system

$$x^+ \in F_{ed}(x) = \{f(x, u) + d \mid u \in \kappa_N(x + e)\} \quad (8)$$

in which $d \in \mathbb{R}^n$ is an *unknown* process disturbance and $e \in \mathbb{R}^n$ represents an *unknown* state measurement/estimate error. (We have used the definition of $\kappa_N(\cdot)$ from Section 2.2, so that $\kappa_N(x+e)$ is the first component of a *suboptimal* solution of $\mathbb{P}_N(x + e)$.)

Remark 20. In the perturbed case, the control sequence $\mathbf{u}$ is computed as a suboptimal solution of $\mathbb{P}_N(x_m)$, with $x_m = x+e$, i.e., it is based on the evolution of nominal system (1), for the initial measured state.

We denote with $\phi_{ed}(k; x) = x(k)$ *a solution* to the perturbed closed-loop system (8) for the initial state $x(0) = x$ and given disturbance and measurement error sequences $\{d(k)\}, \{e(k)\}$. We now present the definition of *robust exponential stability* (RES), similar to that of *robust asymptotic stability* (RAS) given in [12].

Definition 21 (*RES*). The origin of the closed-loop system (8) is *robustly exponentially stable* (RES) on $\mathrm{int}(\mathcal{X}_N)$ if there exist scalars $b > 0$ and $0 < \lambda < 1$ such that for all compact sets $\mathcal{C} \subset \mathcal{X}_N$, with $0 \in \mathrm{int}(\mathcal{C})$, the following property holds: Given any $\epsilon > 0$, there exists $\delta > 0$ such that for all sequences $\{d(k)\}$ and $\{e(k)\}$ with $x(0) = x \in \mathcal{C}$ satisfying:

$$\max_{k\ge 0} |d(k)| \le \delta, \qquad \max_{k\ge 0} |e(k)| \le \delta$$

$$x_m(k) = x(k) + e(k) \in \mathcal{X}_N, \quad x(k) \in \mathcal{X}_N, \text{ for all } k \in \mathbb{I}_{\ge 0},$$

it follows that

$$|\phi_{ed}(k;x)| \leq b\lambda^k |x| + \epsilon, \quad \text{for all } k \in \mathbb{I}_{\geq 0}. \tag{9}$$

Remark 22. In RES (or RAS) the robust stability condition (9) is presented for those (if any) initial states, disturbance and measurement error sequences that a priori ensure feasibility of the perturbed closed-loop trajectories.

The next definition instead requires that feasibility is satisfied at all times for *all sufficiently small* disturbance and measurement error sequences and all initial states in a given compact subset of $\text{int}(\mathcal{X}_N)$.

Definition 23 (*SRES*)**.** The origin of the closed-loop system (8) is *strongly robustly exponentially stable* (SRES) on a compact set $\mathcal{C} \subset \mathcal{X}_N$, $0 \in \text{int}(\mathcal{C})$, if there exist scalars $b > 0$ and $0 < \lambda < 1$ such that the following property holds: Given any $\epsilon > 0$, there exists $\delta > 0$ such that for all sequences $\{d(k)\}$ and $\{e(k)\}$ satisfying

$$|d(k)| \leq \delta \quad \text{and} \quad |e(k)| \leq \delta \quad \text{for all } k \in \mathbb{I}_{\geq 0},$$

and all $x \in \mathcal{C}$, we have that

$$x_m(k) = x(k) + e(k) \in \mathcal{X}_N, \quad x(k) \in \mathcal{X}_N, \quad \text{for all } k \in \mathbb{I}_{\geq 0}, \tag{10a}$$

$$|\phi_{ed}(k;x)| \leq b\lambda^k |x| + \epsilon, \quad \text{for all } k \in \mathbb{I}_{\geq 0}. \tag{10b}$$

Remark 24. The main import of the SRES definition is that the closed-loop system is input-to-state stable (ISS) [19] considering the disturbances (d, e) as the input. For this reason, some researchers substitute the statement "the system is ISS in $\mathcal{C}$" for what we here defined as SRES on $\mathcal{C}$. Since we require an additional restriction on the measurement $x_m(k)$, which is not part of the standard ISS definition, we prefer to use a separate SRES definition in this paper.

4.2. Feasibility issues

Before presenting the robust stability results, we observe that although the warm start $\tilde{\mathbf{u}}$ is feasible for the *predicted* successor state $\tilde{x}^+ = f(x_m, u(0;x_m))$, i.e., $(\tilde{x}^+, \tilde{\mathbf{u}}) \in \mathbb{Z}_N$, it may not be feasible for the *measured* successor state, i.e., $x_m^+ = f(x, u(0;x_m)) + d + e^+$. (The *true* successor state, which is unknown in general, is $x^+ = f(x, u(0;x_m)) + d$.) If $(x_m^+, \tilde{\mathbf{u}}) \notin \mathbb{Z}_N$, the right-hand side of the cost inequality (5b) is not meaningful. In such cases, we need to modify the warm start with a term $\mathbf{p}$ such that $(x_m^+, \tilde{\mathbf{u}} + \mathbf{p}) \in \mathbb{Z}_N$, and to this aim we consider the following additional assumption.

Assumption 25. For any $x, x' \in \mathcal{X}_N$ and $\mathbf{u} \in \mathcal{U}_N(x)$, there exists $\mathbf{u}' \in \mathcal{U}_N(x')$ such that $|\mathbf{u} - \mathbf{u}'| \leq \sigma(|x - x'|)$ for some $\mathcal{K}$-function $\sigma(\cdot)$.

Remark 26. Assumption 25 has been shown to hold, e.g., for linear systems s.t. polytopic constraints on $(x, \mathbf{u})$, and for nonlinear systems without state (or mixed) and terminal constraints.

Remark 27. Assumption 25 also implies that $V_N^0(\cdot)$ is continuous by applying Theorem C.28 in [2].

Among various options for finding $\mathbf{p}$, we consider the following *feasibility* problem (notice that $\tilde{x}^+$ is known):

$$\text{Find } \mathbf{p} \text{ s.t. } \tilde{\mathbf{u}} + \mathbf{p} \in \mathcal{U}_N(x_m^+) \quad \text{and} \quad |\mathbf{p}| \leq \sigma(|x_m^+ - \tilde{x}^+|). \tag{11}$$

Remark 28. If $\tilde{\mathbf{u}} \in \mathcal{U}_N(x_m^+)$, it immediately follows that $\mathbf{p} = 0$ satisfies the feasibility problem (11), and hence Assumption 25 is unnecessary. Furthermore, we do not require Assumption 25 when treating the case without state and terminal constraints in Section 5.

Proposition 29. *Under Assumption 25, for any $(\tilde{x}^+, \tilde{\mathbf{u}}) \in \mathbb{Z}_N$ and $x_m^+ \in \mathcal{X}_N$, the set of solutions to (11) is nonempty.*

Proof. The result follows directly from Assumption 25 by noticing that $\tilde{\mathbf{u}} \in \mathcal{U}_N(\tilde{x}^+)$ and $\tilde{\mathbf{u}} + \mathbf{p} \in \mathcal{U}_N(x_m^+)$. □

Given any $\mathbf{p}$ satisfying (11), and for any given $x_m^+ \in \mathcal{X}_N$, we replace conditions (5) with the following:

$$\mathbf{u}^+ \in \mathcal{U}_N(x_m^+) \tag{12a}$$

$$V_N(x_m^+, \mathbf{u}^+) \leq V_N(x_m^+, \tilde{\mathbf{u}} + \mathbf{p}) \tag{12b}$$

$$V_N(x_m^+, \mathbf{u}^+) \leq V_f(x_m^+) \quad \text{when } x_m^+ \in r\mathbb{B}. \tag{12c}$$

In the perturbed case, the extended state is $z = (x, \mathbf{u})$, where $\mathbf{u}$ is a suboptimal solution to $\mathbb{P}_N(x_m)$ where $x_m := x + e$ is the measured state. The extended system evolves as follows:

$$z^+ \in H_{ed}(z) := \{(x^+, \mathbf{u}^+) \mid x^+ = f(x, u(0;x_m)) + d, \mathbf{u}^+ \in G_{ed}(z)\} \tag{13}$$

in which (notice that both $x_m^+ := x^+ + e^+$ and $\tilde{\mathbf{u}} + \mathbf{p}$ depend on z):

$$G_{ed}(z) := \{\mathbf{u}^+ \mid \mathbf{u}^+ \in \mathcal{U}_N(x_m^+), V_N(x_m^+, \mathbf{u}^+) \leq V_N(x_m^+, \tilde{\mathbf{u}} + \mathbf{p}), V_N(x_m^+, \mathbf{u}^+) \leq V_f(x_m^+) \text{ if } x_m^+ \in r\mathbb{B}\}.$$

4.3. Main results

We denote by $z_m := (x_m, \mathbf{u}) = (x + e, \mathbf{u}) = z + (e, 0)$, we observe that $z_m \in \mathcal{Z}_r$, and present the following supporting result.

Lemma 30. *For every $\mu > 0$, there exists a $\delta > 0$ such that, for all $(z_m, e, d, e^+) \in \mathcal{Z}_r \times \delta\mathbb{B} \times \delta\mathbb{B} \times \delta\mathbb{B}$, $z = z_m - (e, 0)$, such that $x_m^+ \in \mathcal{X}_N$, and some γ, $0 < \gamma < 1$, we have:*

$$\max_{z^+ \in H_{ed}(z)} V_N(z^+) \leq \max\{\gamma V_N(z), \mu\}.$$

Proof. Let $\mu > 0$ be given. The value $V_N(\tilde{x}^+, \tilde{\mathbf{u}})$ is the cost along the nominal trajectory (no disturbance). Therefore since $V_N(\cdot)$ is an exponential Lyapunov function for the nominal system (Lemma 17), Proposition 14 gives that

$$V_N(\tilde{x}^+, \tilde{\mathbf{u}}) \leq V_N(z_m) - \ell(x_m, u(0;x_m)) \leq \bar{\gamma} V_N(z_m)$$

for some $0 < \bar{\gamma} < 1$. Consider a γ such that $\bar{\gamma} < \gamma < 1$, and define $\rho := \mu(\gamma - \bar{\gamma}) > 0$. Recall that: $\tilde{x}^+ - x_m^+ = f(x_m, u(0;x_m)) - f(x, u(0;x_m)) - d - e^+$. Due to continuity of $V_N(\cdot)$ and $f(\cdot)$, and because of $|\mathbf{p}| \leq \sigma(|\tilde{x}^+ - x_m^+|)$, we can choose $\delta_1 > 0$ such that the following condition holds for all $(z_m, e, d, e^+) \in \mathcal{Z}_r \times \delta_1\mathbb{B} \times \delta_1\mathbb{B} \times \delta_1\mathbb{B}$, $z = z_m - (e, 0)$:

$$V_N(x_m^+, \tilde{\mathbf{u}} + \mathbf{p}) \leq V_N(\tilde{x}^+, \tilde{\mathbf{u}}) + \frac{\rho}{3}. \tag{14}$$

By continuity of $V_N(\cdot)$, choose $\delta_2 > 0$ such that the condition:

$$V_N(\tilde{x}^+, \tilde{\mathbf{u}}) \leq \bar{\gamma} V_N(x_m, \mathbf{u}) \leq \bar{\gamma} V_N(x, \mathbf{u}) + \frac{\rho}{3} \tag{15}$$

holds for all $(z_m, e) \in \mathcal{Z}_r \times \delta_2\mathbb{B}$, $z = z_m - (e, 0)$. From continuity of $V_N(\cdot)$ and $f(\cdot)$ and from (12b), choose $\delta_3 > 0$ such that

$$V_N(x^+, \mathbf{u}^+) \leq V_N(x_m^+, \mathbf{u}^+) + \frac{\rho}{3} \leq V_N(x_m^+, \tilde{\mathbf{u}} + \mathbf{p}) + \frac{\rho}{3} \tag{16}$$

for all $z^+ := (x^+, \mathbf{u}^+) \in H_{ed}(z)$ and all $(z_m, e, d, e^+) \in \mathcal{Z}_r \times \delta_3\mathbb{B} \times \delta_3\mathbb{B} \times \delta_3\mathbb{B}$, $z = z_m - (e, 0)$. Defining $\delta := \min\{\delta_1, \delta_1, \delta_3\}$, and summing up (the most external sides of) (14)–(16), we obtain:

$$\max_{z^+ \in H_{ed}(z)} V_N(z^+) \leq \bar{\gamma} V_N(z) + \rho$$

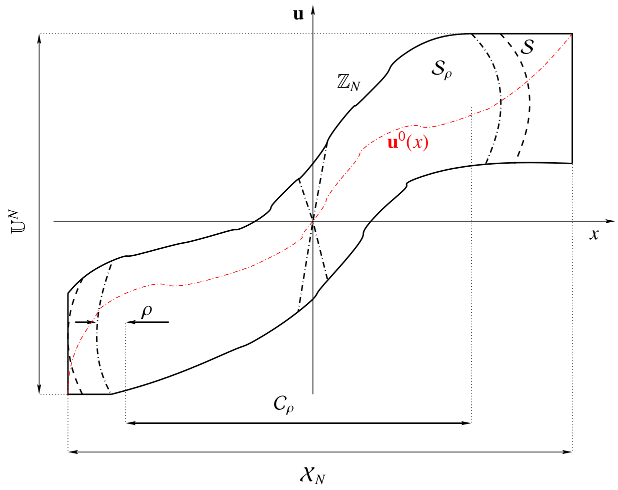

Fig. 1. Sketch of the main sets involved in SRES.

for all $(z_m, e, d, e^+) \in \mathcal{Z}_r \times \delta\mathbb{B} \times \delta\mathbb{B} \times \delta\mathbb{B}, z = z_m - (e, 0)$. Define $\mathcal{Z}_1 := \{z = z_m - (e, 0) \mid z_m \in \mathcal{Z}_r, e \in \delta\mathbb{B}, V_N(z) \le \mu\}$ and $\mathcal{Z}_2 := \{z = z_m - (e, 0) \mid z_m \in \mathcal{Z}_r, e \in \delta\mathbb{B}, V_N(z) > \mu\}$, and assume that μ is not so large that $\mathcal{Z}_2$ is empty (otherwise the proof is simpler). If $z \in \mathcal{Z}_1$ we can write: $\max_{z^+ \in H_{ed}(z)} V_N(z^+) \le \bar{\gamma} V_N(z) + \rho \le \bar{\gamma}\mu + \mu(\gamma - \bar{\gamma}) \le \mu$. If instead $z \in \mathcal{Z}_2$ we can write: $\max_{z^+ \in H_{ed}(z)} V_N(z^+) \le \bar{\gamma} V_N(z) + \mu(\gamma - \bar{\gamma}) \le \gamma V_N(z)$. Therefore, we have established that the condition:

$$\max_{z^+ \in H_{ed}(z)} V_N(z^+) \le \max\{\gamma V_N(z), \mu\}$$

holds for all $(z_m, e, d, e^+) \in \mathcal{Z}_r \times \delta\mathbb{B} \times \delta\mathbb{B} \times \delta\mathbb{B}, z = z_m - (e, 0)$, such that $x_m^+ \in \mathcal{X}_N$. $\square$

We now characterize the compact sets over which SRES is guaranteed to hold. Consider a scalar $\bar{V} > 0$ such that the set:

$$\mathcal{S} := \{z \in \mathbb{R}^n \times \mathbb{U}^N \mid V_N(z) \le \bar{V}\}$$

satisfies $\mathcal{S} \subseteq \mathbb{Z}_N$, i.e., $\mathcal{S}$ is a sublevel set of $\mathbb{R}^n \times \mathbb{U}^N$ fully contained in $\mathbb{Z}_N$. Thus, by definition, for any $z := (x, \mathbf{u}) \in \mathcal{S}$, it follows that $x \in \mathcal{X}_N$. Next, given a scalar $\rho > 0$ and any $z_m \in \mathcal{Z}_r$, we define the following measure and associated set:

$$V_N^\rho(z_m) := \max_{e \in \rho\mathbb{B}} V_N(z) \quad \text{s.t. } z = z_m - (e, 0) \tag{17}$$

$$\mathcal{S}_\rho := \{z_m \in \mathcal{Z}_r \mid V_N^\rho(z_m) \le \bar{V}\} \tag{18}$$

in which we assume that ρ is small enough that $\mathcal{S}_\rho$ is nonempty. Finally, we define the following compact set:

$$\mathcal{C}_\rho := \{x \in \mathbb{R}^n \mid x = x_m - e, e \in \rho\mathbb{B}, \exists \mathbf{u} \text{ s.t. } (x_m, \mathbf{u}) \in \mathcal{S}_\rho\} \tag{19}$$

and we observe that $0 \in \text{int}(\mathcal{C}_\rho) \subset \mathcal{X}_N$ for ρ sufficiently small. These sets are depicted in Fig. 1. The main SRES result of this paper is as follows.

Theorem 31 (*SRES of Suboptimal MPC*). *Under Assumptions 1, 2(a), 3, 4 and 25, the origin of the perturbed closed-loop system (8) is SRES on $\mathcal{C}_\rho$.*

Proof (*Robust Feasibility*). Suppose that $x \in \mathcal{C}_\rho$ and let $z := (x, \mathbf{u})$ be the corresponding augmented state where $\mathbf{u}$ is a suboptimal sequence computed for the measured state $x_m := x + e, e \in \rho\mathbb{B}$. We recall that $V_N(z) \le \bar{V}$, i.e., $z \in \mathcal{S}$ and that $z_m \in \mathcal{S}_\rho \subseteq \mathcal{Z}_r$. Moreover, we define $\tilde{z}^+ := (\tilde{x}^+, \tilde{\mathbf{u}})$. Since $V_N(\cdot)$ is an exponential Lyapunov function for the nominal system and $z_m \in \mathcal{Z}_r$, Proposition 14 gives that $V_N(\tilde{z}^+) \le \bar{\gamma} V_N(z_m)$ for some $0 < \bar{\gamma} < 1$. Because $z_m \in \mathcal{S}_\rho$ it follows that $V_N(\tilde{x}^+, \tilde{\mathbf{u}}) \le \bar{\gamma}\bar{V} < \bar{V}$. Recalling that: $\tilde{x}^+ - x_m^+ = f(x_m, u(0; x_m)) - f(x, u(0; x_m)) - d - e^+$, it follows from continuity of $f(\cdot)$ that there exists a $\bar{\delta}_1 > 0$ such that $V_N(x_m^+, \tilde{\mathbf{u}}) < \bar{V}$ and thus $x_m^+ \in \mathcal{X}_N$ for all $(z_m, e, d, e^+) \in \mathcal{S}_\rho \times \bar{\delta}_1 \times \bar{\delta}_1 \times \bar{\delta}_1$. Hence, the initialization step (11) is well defined. Define any $0 < \mu < (1 - \bar{\gamma})\bar{V}$. From continuity of $V_N(\cdot)$ and $f(\cdot)$, and because $|\mathbf{p}| \le \sigma(|\tilde{x}^+ - x_m^+|)$, we can choose $\bar{\delta}_2 > 0$ such that the following condition holds: $V_N(x_m^+, \tilde{\mathbf{u}} + \mathbf{p}) \le V_N(\tilde{x}^+, \tilde{\mathbf{u}}) + \mu < V_N(\tilde{x}^+, \tilde{\mathbf{u}}) + (1 - \bar{\gamma})\bar{V} \le \bar{V}$. This proves that $V_N(z_m^+) \le V_N(x_m^+, \tilde{\mathbf{u}} + \mathbf{p}) < \bar{V}$. From continuity of $V_N(\cdot)$, it also follows that we can choose $\rho > 0$ sufficiently small that $V_N^\rho(z_m^+) \le \bar{V}$. Taking $\delta = \min\{\rho, \bar{\delta}_1, \bar{\delta}_2\}$ we have proved that $z_m^+ \in \mathcal{S}_\rho$ for all $(z_m, e, d, e^+) \in \mathcal{S}_\rho \times \delta\mathbb{B} \times \delta\mathbb{B} \times \delta\mathbb{B}$. This implies:

$$x(k) \in \mathcal{C}_\rho \subseteq \mathcal{X}_N \quad \text{for all } k \in \mathbb{I}_{\ge 0}$$

and also that $x_m(k) \in \mathcal{X}_N$ for all $k \in \mathbb{I}_{\ge 0}$. Hence, (10a) holds.

(*Robust Stability*). We denote by $\psi_{ed}(k; z)$ *a solution* of perturbed difference inclusion (13) at time $k \in \mathbb{I}_{\ge 0}$ starting from the initial state $z(0) = z$ and given disturbance and measurement error sequences $\{d(k)\}, \{e(k)\}$. As established in the proof of Proposition 11, we have that there exists a scalar $a_1 > 0$ such that $a_1|z|^a \le V_N(z)$ for any $z \in \mathcal{C}_\rho \times \mathbb{U}^N \subseteq \mathbb{X} \times \mathbb{U}^N$. Moreover, by Lemma 17, there exists a scalar $a_2 > 0$ such that $V_N(z) \le a_2|z|^a$ for any $z \in \mathcal{C}_\rho \times \mathbb{U}^N$. From Lemma 30, by induction, we now write:

$$\begin{aligned} a_1|\psi_{ed}(k; z)|^a &\le V_N(\psi_{ed}(k; z)) \\ &\le \max\{\gamma^k V_N(z), \mu\} \le \max\{\gamma^k a_2|z|^a, \mu\} \end{aligned}$$

which implies

$$|\psi_{ed}(k; z)| \le \max\{\bar{b}\lambda^k|z|, (\mu/a_1)^{1/a}\} \le \max\{\bar{b}\lambda^k|z|, \bar{\epsilon}\}$$

in which $\lambda = \gamma^{1/a}$, $\bar{b} = (a_2/a_1)^{1/a}$ and $\bar{\epsilon} = (\mu/a_1)^{1/a}$. Finally, from Lemma 16 recalling that $|\mathbf{u}| \le c|x_m| \le c|x| + c\delta$ and that $\phi_{ed}(k; x)$ represents the state component of $\psi_{ed}(k; z)$, for all $x \in \mathcal{C}_\rho$, we write:

$$\begin{aligned} |\phi_{ed}(k; x)| &\le |\psi_{ed}(k; z)| \le \max\{\bar{b}\lambda^k|z|, \bar{\epsilon}\} \le \bar{b}\lambda^k|z| + \bar{\epsilon} \\ &\le b\lambda^k|x| + \epsilon \end{aligned}$$

with $b = \bar{b}(1 + c)$ and $\epsilon = \bar{\epsilon} + \bar{b}c\delta$, completing the proof because $0 < \lambda < 1$. $\square$

Corollary 32 (*RES of Suboptimal MPC*). *Under Assumptions 1, 2(a), 3, 4 and 25, the origin of perturbed closed-loop system (8) is RES on int($\mathcal{X}_N$).*

Proof. This result follows immediately because robust feasibility is assumed in RES. Thus, for any compact set $\mathcal{C} \subset \text{int}\,\mathcal{X}_N$, the second part (Robust stability) of the proof of Theorem 31 can be readily applied to obtain for all $x \in \mathcal{C}$:

$$|\phi_{ed}(k; x)| \le b\lambda^k|x| + \epsilon \quad \text{for all } k \in \mathbb{I}_{\ge 0}. \quad \square$$

Remark 33. While RES holds in int($\mathcal{X}_N$), SRES is guaranteed to hold in a compact subset $\mathcal{C}_\rho \subset \text{int}(\mathcal{X}_N)$, $0 \in \text{int}(\mathcal{C}_\rho)$. However, SRES may hold even in larger subsets of int($\mathcal{X}_N$).

4.4. Further comments on robust stability under optimal MPC

We briefly observe that given the result of Proposition 9, all robust stability results that we proved for suboptimal MPC, readily apply to optimal MPC even for cases in which there exist more than one optimal solution point, i.e., when the optimal control $\mathbf{u}^0(\cdot)$ is a set-valued map.

Moreover, for optimal MPC, definition (17) of V_N^ρ (and hence of $\mathcal{S}_\rho$) can be modified as follows:

$$V_N^\rho(x_m, \mathbf{u}^0(x_m)) = \max_{e \in \rho\mathbb{B}} V_N(x_m - e, \mathbf{u}^0(x_m)). \tag{20}$$

5. Robust stability in absence of state constraints

One of the goals in replacing state constraints $x(k) \in \mathbb{X}$ with penalties, and allowing small state constraint violations (soft state constraints), is to enlarge the region of attraction of the origin under process and measurement disturbances. Another benefit is to streamline the required robustness analysis. We will also replace the terminal constraint, $x(N) \in \mathbb{X}_f$, with a suitably adjusted terminal penalty $\beta V_f(x(N))$. But, unlike the softened state constraints, the terminal constraint must be satisfied or one cannot invoke the local control Lyapunov function in the terminal region to ensure stability. So we can regard the choice of terminal penalty weight β not as a method of approximately satisfying the terminal constraint, but as a method of enforcing it exactly for some well-defined set of initial states. We next characterize the region of attraction and provide a streamlined robustness analysis for the case excluding state constraints, $\mathbb{X} = \mathbb{R}^n$, but including the terminal constraint, either explicitly or by using a terminal penalty factor β.

5.1. Revised assumptions and nominal stability results

We now specialize the results on inherent robustness for the case in which there are no state constraints. To this aim, we replace Assumption 2(a) with (b). Moreover, as discussed later on, Assumption 25 will not be necessary, whereas Assumptions 1, 3 and 4 (with $V_N(\cdot)$ replaced by $V_N^\beta(\cdot)$ later defined) are required. We modify the cost function as follows:

$$V_N^\beta(x, \mathbf{u}) = \sum_{k=0}^{N-1} \ell(\phi(k; x, \mathbf{u}), u(k)) + \beta V_f(\phi(N; x, \mathbf{u}))$$

in which $\beta \geq 1$ is a parameter that will be chosen in way that the terminal constraint, $\phi(N; x, \mathbf{u}) \in \mathbb{X}_f$, is unnecessary as it will be satisfied inherently for any suboptimal input sequence with appropriately bounded cost. Given the warm start $\tilde{\mathbf{u}}$ for the successor state $x^+ = f(x, u(0; x))$, defined as in (4), we modify the requirements to the suboptimal MPC algorithm as follows:

$$\mathbf{u}^+ \in \mathbb{U}^N \tag{21a}$$

$$V_N^\beta(x^+, \mathbf{u}^+) \leq V_N^\beta(x^+, \tilde{\mathbf{u}}) \tag{21b}$$

$$V_N^\beta(x^+, \mathbf{u}^+) \leq \beta V_f(x^+) \quad \text{when } x^+ \in r\mathbb{B}. \tag{21c}$$

Remark 34. The main difference between the above requirements and those in (5) is that in (21a), we allow any input $\mathbf{u}^+ \in \mathbb{U}^N$, whereas in (5), the terminal constraint, $\phi(N; x, \mathbf{u}) \in \mathbb{X}_f$, is explicitly enforced by (5a). Condition (21c) is also slightly different and follows from the modification of the terminal penalty.

To avoid unnecessary repetition, we again use (6) (or (7) when referring to the extended state) to describe the evolution of the nominal closed-loop system under suboptimal MPC with modified terminal penalty. We choose scalar (maximal cost) $\bar{V} > 0$ and define the following compact sets:

$$\begin{aligned}\bar{\mathcal{Z}}_r &:= \{(x, \mathbf{u}) \in \mathbb{R}^n \times \mathbb{U}^N \mid V_N^\beta(x, \mathbf{u}) \leq \bar{V}, \\ &\quad\ \text{and } V_N^\beta(x, \mathbf{u}) \leq \beta V_f(x) \text{ if } x \in r\mathbb{B}\} \\ \mathbb{X}_0 &:= \{x \in \mathbb{R}^n \mid \exists \mathbf{u} \in \mathbb{U}^N \text{ such that } (x, \mathbf{u}) \in \bar{\mathcal{Z}}_r\}.\end{aligned} \tag{22}$$

Remark 35. For the remainder of the paper, we choose:

$$\bar{V} \geq \alpha \quad \beta = \bar{\beta} := \bar{V}/\alpha$$

in which $\bar{V}$ is the maximal cost in the previous set definitions and $\alpha > 0$ is the terminal region sublevel set parameter of Assumption 2(b). From the choice of $\bar{V}$, we also have that $\beta \geq 1$, so that we have allowed only an *increase* in the terminal penalty. Note that all the results to follow also hold if we choose any β satisfying $\beta \geq \bar{\beta}$.

Remark 36. The choice $\beta \geq \bar{\beta}$ also implies that $\bar{\mathcal{Z}}_r$ does *not* contain any trajectories terminating on the boundary of $\mathbb{X}_f$. For such trajectories, $V_f(x(N)) = \alpha$, and thus $\sum_{i=0}^{N-1} \ell(x(i), u(i)) \leq 0$, which is satisfied only by $(x(i), u(i)) = (0, 0)$ for $i \in \mathbb{I}_{0:N-1}$, which implies that $x(N) = 0$, which is a contradiction. Therefore, $\mathbb{X}_0$ contains only states that can be steered to $\text{int}(\mathbb{X}_f)$.

We have the following nominal stability results.

Lemma 37. *$V_N^\beta(\cdot)$ is an exponential Lyapunov function for extended closed-loop system (7) in any compact subset of $\bar{\mathcal{Z}}_r$.*

Proof. Following the proof of Lemma 16, we have again that there exists a constant c such that for any $(x, \mathbf{u}) \in \bar{\mathcal{Z}}_r$, the condition $|\mathbf{u}| \leq c|x|$ holds. Now consider the difference inclusion $z^+ \in H(z)$, where $H(\cdot)$ is defined in (7) with $G(\cdot)$ appropriately modified according to (21). Following the proof of Lemma 17, it follows again that for any $z = (x, \mathbf{u}) \in \bar{\mathcal{Z}}_r$ the conditions:

$$a_1|z|^a \leq V_N^\beta(z) \leq a_2|z|^a, \qquad \max_{z^+ \in H(z)} V_N^\beta(z^+) \leq V_N^\beta(z) - a_3|z|^a$$

hold for some positive scalars a_1, a_2, a_3. Hence, $V_N^\beta(\cdot)$ is an exponential Lyapunov function on the set $\bar{\mathcal{Z}}_r$ for (7). □

Theorem 38 (*ES Without State Constraints*)**.** *Under Assumptions 1, 2(b), 3 and 4, the origin of closed-loop system (6) is ES on $\mathbb{X}_0$.*

Proof. We first show that $\bar{\mathcal{Z}}_r$ is positively invariant for $z^+ \in H(z)$. To this aim, assume that $z \in \bar{\mathcal{Z}}_r$ and consider any $z^+ \in H(z)$. From (21b), it follows that:

$$\begin{aligned}V_N^\beta(z^+) &:= \sum_{k=0}^{N-1} \ell(\phi(k; x^+, \mathbf{u}^+)) + \beta V_f(\phi(N; x^+, \mathbf{u}^+)) \\ &\leq V_N^\beta(x^+, \tilde{\mathbf{u}}) \leq V_N^\beta(z) \leq \bar{V}.\end{aligned}$$

Since $\beta := \max\{1, \bar{V}/\alpha\}$ and by nonnegativeness of $\ell(\cdot)$, it follows that $V_f(\phi(N; x^+, \mathbf{u}^+)) \leq \alpha$, i.e., $\phi(N; x^+, \mathbf{u}^+) \in \mathbb{X}_f$. Combining this, (21a) and (21c), it follows that $z^+ \in \bar{\mathcal{Z}}_r$. Thus, from Lemmas 37 and 15, the origin of extended closed-loop system (7) is ES on $\bar{\mathcal{Z}}_r$, i.e., all solutions $\psi(k; z)$ satisfy (for some $b' > 0$ and $0 < \lambda < 1$) the conditions: $\psi(k; z) \in \bar{\mathcal{Z}}_r$ and $|\psi(k; z)| \leq b'\lambda^k|z|$ at all times $k \geq \mathbb{I}_{\geq 0}$. Finally, for any $x \in \mathbb{X}_0$ and any input sequence $\mathbf{u}$ such that $(x, \mathbf{u}) \in \bar{\mathcal{Z}}_r$, let $\phi(k; x)$ denote the state component of $\psi(k; z)$. It follows that, at all times $k \in \mathbb{I}_{\geq 0}$, the conditions, $\phi(k; x) \in \mathbb{X}_0$ and $|\phi(k; x)| \leq |\psi(k; z)| \leq b'\lambda^k|z| \leq b\lambda^k|x|$, hold with $b = b'(1 + c)$. □

Next we would like to characterize the newly introduced admissible set $\mathbb{X}_0$ and its limit for large $\bar{V}$. To this end, we define a (slightly) restricted feasible set of initial states that can be taken by an admissible input sequence to the *interior* of $\mathbb{X}_f$, rather than all of $\mathbb{X}_f$ (note that the interior of $\mathbb{X}_f$ is not empty because $\alpha > 0$):

$$\bar{\mathcal{X}}_N := \{x \in \mathbb{R}^n \mid \exists \mathbf{u} \in \mathbb{U}^N \text{ such that } \phi(N; x, \mathbf{u}) \in \text{int}(\mathbb{X}_f)\}. \tag{23}$$

We have the following result.

Proposition 39 (*Admissible Set $\mathbf{X}_0$ and Feasible Set $\bar{\mathcal{X}}_N$*)**.** *The admissible set $\mathbb{X}_0$ and restricted feasible set $\bar{\mathcal{X}}_N$ satisfy the following:*

$$\mathbb{X}_0(\bar{V}) \subseteq \bar{\mathcal{X}}_N \quad \text{for all } \bar{V} \geq 0, \quad \text{and} \quad \bar{\mathcal{X}}_N \subseteq \bigcup_{\bar{V} \geq 0} \mathbb{X}_0(\bar{V}). \tag{24}$$

Proof. The fact that $\mathbb{X}_0(\bar{V}) \subseteq \bar{\mathcal{X}}_N$ for all $\bar{V} \geq 0$ follows directly from definitions (23) and Remark 36: $\bar{\mathcal{X}}_N$ is the set of states that can be brought to the interior of $\mathbb{X}_f$ with feasible inputs, and $\mathbb{X}_0(\bar{V})$ is the set that can be brought to the interior of $\mathbb{X}_f$ with feasible inputs *and* cost not exceeding $\bar{V}$.

We next establish the second inclusion. First, we show that the sets $\mathbb{X}_0(\bar{V})$ are nested: $\bar{V}_2 \geq \bar{V}_1$ implies $\mathbb{X}_0(\bar{V}_2) \supseteq \mathbb{X}_0(\bar{V}_1)$. Assume an arbitrary $x \in \mathbb{X}_0(\bar{V}_1)$, and corresponding $(x, \mathbf{u}) \in \bar{\mathcal{Z}}_r(\bar{V}_1)$. We show that $x \in \mathbb{X}_0(\bar{V}_2)$. Let $\beta_1 := \bar{V}_1/\alpha$, $\beta_2 := \bar{V}_2/\alpha$ and $x(N) := \phi(N; x, \mathbf{u})$. We have that:

$$V_N^{\beta_2}(x, \mathbf{u}) = V_N^{\beta_1}(x, \mathbf{u}) + (\beta_2 - \beta_1)V_f(x(N)).$$

First, notice that if $x \in r\mathbb{B}$, $V_N^{\beta_1}(x, \mathbf{u}) \leq \beta_1 V_f(x)$, and this implies that $V_N^{\beta_2}(x, \mathbf{u}) \leq \beta_2 V_f(x)$, so the required inequality is established in $r\mathbb{B}$. Then notice that $V_f(x(N)) = \alpha' < \alpha$, which gives

$$\begin{aligned} V_N^{\beta_2}(x, \mathbf{u}) &\leq \bar{V}_1 + (\bar{V}_2 - \bar{V}_1)(\alpha'/\alpha) = \bar{V}_1(1 - \alpha'/\alpha) + \bar{V}_2(\alpha'/\alpha) \\ &\leq \bar{V}_2(1 - \alpha'/\alpha) + \bar{V}_2(\alpha'/\alpha) = \bar{V}_2 \end{aligned}$$

and we conclude $x \in \mathbb{X}_0(\bar{V}_2)$. Next, we establish that for every point $x_0 \in \bar{\mathcal{X}}_N$, there exists a $\bar{V}_0 > 0$ such that $x_0 \in \mathbb{X}_0(\bar{V})$ for all $\bar{V}$ satisfying $\bar{V} \geq \bar{V}_0$. Take an arbitrary $x_0 \in \bar{\mathcal{X}}_N$ and a corresponding $\mathbf{u}_0 \in \mathbb{U}^N$ that satisfies $\phi(N; x_0, \mathbf{u}_0) \in \text{int}(\mathbb{X}_f)$. If $x_0 \in r\mathbb{B}$, add the restriction to $\mathbf{u}_0$ that $V_N(x_0, \mathbf{u}_0) \leq V_f(x_0)$, where $V_N(\cdot)$ continues to denote the original cost function, i.e., with non-inflated terminal cost. Such a $\mathbf{u}_0$ exists because of Proposition 9, which establishes that the optimal input sequence, for example, has this property. And since $\beta \geq 1$, it follows that $V_N^{\beta}(x_0, \mathbf{u}_0) \leq \beta V_N(x_0, \mathbf{u}_0) \leq \beta V_f(x_0)$, if $x_0 \in r\mathbb{B}$. Then denote by α' the terminal cost $\alpha' := V_f(\phi(N; x_0, \mathbf{u}_0))$, and we have that $\alpha' < \alpha$. Then define $\bar{V}_0 := \left(\frac{1}{1-\alpha'/\alpha}\right) \sum_{i=0}^{N-1} \ell(\phi(i; x_0, \mathbf{u}_0), u_0(i))$. A direct computation gives $V_N^{\beta}(x_0, \mathbf{u}_0) = \bar{V}_0$, and, if $x_0 \in r\mathbb{B}$, $V_N^{\beta}(x_0, \mathbf{u}_0) \leq \beta V_f(x_0)$. Therefore $x_0 \in \mathbb{X}_0(\bar{V}_0)$, and by the nesting property, $x_0 \in \mathbb{X}_0(\bar{V})$ for all $\bar{V}$ satisfying $\bar{V} \geq \bar{V}_0$, and the limit is established. □

Remark 40. Another characterization of the same result is that the limit of $\mathbb{X}_0(\bar{V})$ as $\bar{V} \to \infty$ contains all feasible sets corresponding to (arbitrarily small) tightening of the terminal set, $\mathbb{X}_f(\alpha') := \text{lev}_{\alpha'} V_f$ using $\alpha' < \alpha$.

5.2. Robust stability results

For robustness analysis, we again consider that the closed-loop system evolves according to (8). We observe that having removed the terminal constraint has the immediate consequence that the warm start $\tilde{\mathbf{u}}$ is "feasible" for the measured successor state $x_m^+ := x^+ + e^+$, because $\tilde{\mathbf{u}} \in \mathbb{U}^N$. Hence, there is no need to solve feasibility problem (11). This is the main motivation for removing the terminal constraint and inflating the terminal cost. Therefore, we can write the evolution of the extended closed-loop system as $z^+ \in H_{ed}(z)$ in which $H_{ed}(\cdot)$ is still defined in (13) with $G_{ed}(\cdot)$ modified as follows:

$$\begin{aligned} G_{ed}(z) := \{\mathbf{u}^+ \mid \mathbf{u}^+ \in \mathbb{U}^N, V_N^{\beta}(x_m^+, \mathbf{u}^+) \leq V_N^{\beta}(x_m^+, \tilde{\mathbf{u}}), \\ V_N^{\beta}(x_m^+, \mathbf{u}^+) \leq \beta V_f(x_m^+) \text{ if } x_m^+ \in r\mathbb{B}\}. \end{aligned}$$

We also observe that the fundamental result of Lemma 30 still holds for the modified cost $V_N^{\beta}(\cdot)$, with $\mathcal{Z}_r$ replaced by $\bar{\mathcal{Z}}_r$.

We now present a set over which we prove SRES. To this aim, given a scalar $\rho > 0$ and any $z_m \in \bar{\mathcal{Z}}_r$, we define:

$$\bar{V}_N^{\rho}(z_m) := \max_{e \in \rho\mathbb{B}} V_N^{\beta}(z) \quad \text{s.t. } z = z_m - (e, 0) \tag{25a}$$

$$\bar{\mathcal{S}}_{\rho} := \{z_m \in \bar{\mathcal{Z}}_r \mid \bar{V}_N^{\rho}(z_m) \leq \bar{V}\} \tag{25b}$$

in which we assume that ρ is small enough that $\bar{\mathcal{S}}_{\rho}$ is nonempty. Finally, the candidate set for SRES is defined as:

$$\bar{\mathcal{C}}_{\rho} := \{x \in \mathbb{R}^n \mid x = x_m - e, e \in \rho\mathbb{B}, \exists \mathbf{u} \text{ s.t. } (x_m, \mathbf{u}) \in \bar{\mathcal{S}}_{\rho}\}. \tag{26}$$

Theorem 41 (*SRES Without State Constraints*). *Under Assumptions* 1, 2(b), 3 *and* 4, *the origin of closed-loop system* (8) *is SRES on* $\bar{\mathcal{C}}_{\rho}$.

Proof (*Robust Feasibility*). Suppose that $x \in \bar{\mathcal{C}}_{\rho}$ and let $z := (x, \mathbf{u})$ be the corresponding augmented state where $\mathbf{u}$ is a suboptimal sequence computed for the measured state $x_m := x + e$, $e \in \rho\mathbb{B}$. We recall that from (25), it follows that $V_N^{\beta}(z) \leq \bar{V}$ and that $z_m \in \bar{\mathcal{S}}_{\rho} \subseteq \bar{\mathcal{Z}}_r$. Moreover, we define $\tilde{z}^+ := (\tilde{x}^+, \tilde{\mathbf{u}})$ where we recall that $\tilde{x}^+ = f(x_m, u(0; x_m))$ is the successor state for the nominal system (no disturbance). The quantity $V_N^{\beta}(\tilde{z}^+)$ is the cost along the nominal evolution from z_m (no disturbance). Since $V_N^{\beta}(\cdot)$ is an exponential Lyapunov function for the nominal system on set $\bar{\mathcal{Z}}_r$ (Lemma 37), we have from Proposition 14 that $V_N^{\beta}(\tilde{z}^+) \leq \bar{\gamma} V_N^{\beta}(z_m)$ for some $0 < \bar{\gamma} < 1$. Thus, it follows that $V_N^{\beta}(\tilde{x}^+, \tilde{\mathbf{u}}) \leq \bar{\gamma}\bar{V} < \bar{V}$. Recalling that: $\tilde{x}^+ - x_m^+ = f(x_m, u(0; x_m)) - f(x, u(0; x_m)) - d - e^+$, it follows from continuity of $V_N^{\beta}(\cdot)$ that there exists a $\tilde{\delta}_1 > 0$ such that: $V_N^{\beta}(x_m^+, \tilde{\mathbf{u}}) < \bar{V}$ holds (strictly) for all $(z_m, e, d, e^+) \in \bar{\mathcal{S}}_{\rho} \times \tilde{\delta}_1\mathbb{B} \times \tilde{\delta}_1\mathbb{B} \times \tilde{\delta}_1\mathbb{B}$. Since $\beta = \max\{1, \bar{V}/\alpha\}$, from (21b) and given that $V_N^{\beta}(x_m^+, \mathbf{u}^+) := \sum_{k=0}^{N-1} \ell(\phi(k; x_m^+, \mathbf{u}^+)) + \beta V_f(\phi(N; x_m^+, \mathbf{u}^+))$, it follows that $V_f(\phi(N; x_m^+, \mathbf{u}^+)) < \alpha$, which proves that $\phi(N; x_m^+, \mathbf{u}^+) \in \text{int}(\mathbb{X}_f)$. From continuity of $V_N^{\beta}(\cdot)$, it also follows that we can choose $\rho > 0$ sufficiently small that $\bar{V}_N^{\rho}(z_m^+) \leq \bar{V}$. Taking $\delta = \min\{\rho, \tilde{\delta}_1\}$, we have proved that $z_m^+ \in \bar{\mathcal{S}}_{\rho}$ for all $(z_m, e, d, e^+) \in \bar{\mathcal{S}}_{\rho} \times \delta\mathbb{B} \times \delta\mathbb{B} \times \delta\mathbb{B}$. This implies:

$$x(k) \in \bar{\mathcal{C}}_{\rho} \subseteq \mathbb{X}_0 \quad \text{for all } k \in \mathbb{I}_{\geq 0}$$

and also that $x_m(k) \in \mathbb{X}_0$ for all $k \in \mathbb{I}_{\geq 0}$. Hence, (10a) holds (with $\mathcal{X}_N$ replaced by $\mathbb{X}_0$).

(*Robust Stability*). This part follows exactly the corresponding part in the proof of Theorem 31 and is omitted. (Note that feasibility recovery step (11) is not mentioned in the robust stability part of the proof of Theorem 31.) □

Finally, we would like to characterize the robust region of attraction $\bar{\mathcal{C}}_{\rho}$. The ideal, but unachievable, result would be that the robust region of attraction under nonzero disturbances is the entire nominal feasible set, $\mathcal{X}_N$. We see next, however, that this ideal result is approached reasonably closely when excluding state constraints.

Remark 42. When $|d|, |e| \to 0$, it follows directly from (25) and (26) that $\bar{\mathcal{C}}_{\rho} \to \mathbb{X}_0$ and SRES holds over a set approaching the admissible set of initial conditions.

This observation, coupled with (24), gives the desired result: in the limit of small disturbances and large parameter $\bar{V}$, the robust region of attraction for the case without state constraints converges to (the closure of) the restricted feasible set.

6. Conclusions

Solving MPC problems for nonlinear systems *globally* is not practical in most cases, and when *a suboptimal* solution is implemented in closed loop, many stability questions arise. The paper [4] proved nominal asymptotic stability of the origin of the closed-loop system in a neighborhood of the origin. In this paper, under similar assumptions, we went several steps further. First, we established nominal *exponential* stability of the origin of the closed-loop system in arbitrarily large compact subsets of the feasible set, i.e., the set of initial states for which a feasible solution exists (if this set is itself compact, then exponential stability is established on the entire set). Second, and more importantly, we established *inherent exponential robust* stability of the origin of the closed-loop system, under general and implementable suboptimal MPC, with respect to additive process disturbances and measurement errors, in the spirit of the ideas developed by Teel and coworkers [12,13]. In this paper, we *proved* robust recursive feasibility in an appropriate compact subset of the nominal region of attraction for all, sufficiently small, but arbitrary, disturbances. All the results that we established here for suboptimal MPC apply also to optimal MPC, and the optimal MPC results also represent improvements in stability analysis of nonlinear MPC systems.

In the absence of state constraints, a variant of the controller is proposed, in which the terminal constraint is replaced by an inflated terminal cost that ensures satisfaction of the terminal constraint. Thus, we proved that nominal exponential stability of the origin under suboptimal MPC holds over a well-defined set of initial conditions. Moreover, we established strong robust exponential stability over a set that approaches the nominal feasibility set when the disturbances go to zero. These general results can be specialized further to suboptimal MPC of *linear* systems, and this specialization is discussed in a separate publication [18].

Given the discussion in the paper, we can now more fully describe the difference between this approach and that of [15]. First, as stated in the introduction, in [15], the user specifies a degree of suboptimality, $\delta > 0$, and the suboptimal MPC cost must satisfy $V_N(x, \mathbf{u}) \leq V_N^0(x) + \delta$. The problems considered here are nonconvex, so we cannot enforce this kind of constraint; we instead compute for each encountered measurement (state) a feasible warm start, and define the suboptimal MPC cost to be no worse than the cost of the warm start. For the case without state constraints, the feasible warm start is available from the last MPC execution, and does not have to be computed. Second, as stated in Remark 5, we allow the input constraints to be active at the origin, which is a common situation in industrial applications. Including this possibility makes the results more relevant to applications, but also increases significantly the complexity of the analysis required to establish robust stability. Paper [15] assumes that no constraints are active in a sufficiently small neighborhood of the origin. Third, we consider both process disturbances and measurement disturbances, and [15] does not consider measurement disturbances. On the other hand, interestingly, [15] treats discontinuous (switching) model dynamics, and we assume $f(\cdot)$ continuous in this paper.

Finally, turning to the last statement of the abstract, the best that currently can be established about the inherent robustness of *optimal* MPC is Theorem 31. By establishing the same result for *suboptimal* MPC, we conclude that there is no qualitative change in robustness when shifting from optimal MPC to suboptimal MPC for the class of models considered here. Certainly, we expect the *size* of the disturbances for which SRES holds to shrink, but the qualitative behavior does not change. This conclusion gives theoretical support to practitioners considering implementing suboptimal MPC with nonlinear plant models and nonconvex MPC optimization problems.

Acknowledgments

The authors would like to thank A.R. Teel for providing an updated and detailed proof of proposition 8 in [12], and D.Q. Mayne and R. Amrit for helpful comments on the content of the paper.

Appendix. Technical lemma

Lemma 43. *Given vectors* $y_i \in \mathbb{R}^{n_i}$, $i = 1, 2, \ldots, N$, *we have for any* $a > 0$ *that* $|(y_1, y_2, \ldots, y_N)|^a \leq N^a \sum_{i=1}^N |y_i|^a$.

Proof. $|(y_1, y_2, \ldots, y_N)|^a \leq \left(\sum_{i=1}^N |y_i|\right)^a \leq \left(N \max_{i=1,\ldots,N} |y_i|\right)^a \leq N^a \max_{i=1,2,\ldots,N} |y_i|^a \leq N^a \sum_{i=1}^N |y_i|^a$. □

References

[1] D.Q. Mayne, J.B. Rawlings, C.V. Rao, P.O.M. Scokaert, Constrained model predictive control: stability and optimality, Automatica 36 (6) (2000) 789–814.
[2] J.B. Rawlings, D.Q. Mayne, Model Predictive Control: Theory and Design, Nob Hill Publishing, Madison, WI, ISBN: 978-0-9759377-0-9, 2009, p. 576.
[3] S.P. Boyd, L. Vandenberghe, Convex Optimization, Cambridge University Press, 2004.
[4] P.O.M. Scokaert, D.Q. Mayne, J.B. Rawlings, Suboptimal model predictive control feasibility implies stability, IEEE Trans. Automat. Control 44 (3) (1999) 648–654.
[5] A. Bemporad, M. Morari, Control of systems integrating logic, dynamics, and constraints, Automatica 35 (1999) 407–427.
[6] D.Q. Mayne, W. Langson, Robustifying model predictive control of constrained linear systems, Electron. Lett. 37 (23) (2001) 1422–1423.
[7] G. Pannocchia, E.C. Kerrigan, Offset-free receding horizon control of constrained linear systems, AIChE J. 51 (2005) 3134–3146.
[8] S. Rakovic, E. Kerrigan, D. Mayne, J. Lygeros, Reachability analysis of discrete-time systems with disturbances, IEEE Trans. Automat. Control 51 (4) (2006) 546–561.
[9] D. Limón Marruedo, T. Álamo, E.F. Camacho, Input-to-state stable MPC for constrained discrete-time nonlinear systems with bounded additive disturbances, in: Proceedings of the 41st IEEE Conference on Decision and Control, Las Vegas, Nevada, 2002, pp. 4619–4624.
[10] G. De Nicolao, L. Magni, R. Scattolini, Stabilizing nonlinear receding horizon control via a nonquadratic penalty, in: Proceedings IMACS Multiconference CESA, Vol. 1, Lille, France, 1996, pp. 185–187.
[11] P.O.M. Scokaert, J.B. Rawlings, E.S. Meadows, Discrete-time stability with perturbations: application to model predictive control, Automatica 33 (3) (1997) 463–470.
[12] G. Grimm, M.J. Messina, S.E. Tuna, A.R. Teel, Examples when nonlinear model predictive control is nonrobust, Automatica 40 (2004) 1729–1738.
[13] G. Grimm, M.J. Messina, S.E. Tuna, A.R. Teel, Nominally robust model predictive control with state constraints, IEEE Trans. Automat. Control 52 (10) (2007) 1856–1870.
[14] M. Lazar, W. Heemels, A.R. Teel, Lyapunov functions, stability and input-to-state stability subtleties for discrete-time discontinuous systems, IEEE Trans. Automat. Control 54 (10) (2009) 2421–2425.
[15] M. Lazar, W. Heemels, Predictive control of hybrid systems: input-to-state stability results for sub-optimal solutions, Automatica 45 (1) (2009) 180–185.
[16] C.M. Kellet, A.R. Teel, Smooth Lyapunov functions and robustness of stability for difference inclusions, Systems Control Lett. 52 (2004) 395–405.
[17] G. Pannocchia, J.B. Rawlings, S.J. Wright, Fast, large-scale model predictive control by partial enumeration, Automatica 43 (2007) 852–860.
[18] G. Pannocchia, S.J. Wright, J.B. Rawlings, Robust stability of suboptimal linear MPC based on partial enumeration, in: DYCOPS 9, J. Proc. Control (November) (2010) (special issue) (in press).
[19] E.D. Sontag, Y. Wang, On the characterization of the input to state stability property, Systems Control Lett. 24 (1995) 351–359.