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Cooperative distributed model predictive control[☆]

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ABSTRACT

In this paper we propose a cooperative distributed linear model predictive control strategy applicable to any finite number of subsystems satisfying a stabilizability condition. The control strategy has the following features: hard input constraints are satisfied; terminating the iteration of the distributed controllers prior to convergence retains closed-loop stability; in the limit of iterating to convergence, the control feedback is plantwide Pareto optimal and equivalent to the centralized control solution; no coordination layer is employed. We provide guidance in how to partition the subsystems within the plant.

We first establish exponential stability of suboptimal model predictive control and show that the proposed cooperative control strategy is in this class. We also establish that under perturbation from a stable state estimator, the origin remains exponentially stable. For plants with sparsely coupled input constraints, we provide an extension in which the decision variable space of each suboptimization is augmented to achieve Pareto optimality. We conclude with a simple example showing the performance advantage of cooperative control compared to noncooperative and decentralized control strategies.

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1. Introduction

Model predictive control (MPC) has been widely adopted in the petrochemical industry for controlling large, multi-variable processes. MPC solves an online optimization to determine inputs, taking into account the current conditions of the plant, any disturbances affecting operation, and imposed safety and physical constraints. Over the last several decades, MPC technology has reached a mature stage. Closed-loop properties are well understood [1], and nominal stability has been demonstrated for many industrial applications [2].

Chemical plants usually consist of linked unit operations and can be subdivided into a number of subsystems. These subsystems are connected through a network of material, energy, and information streams. Because plants often take advantage of the economic savings available in material recycle and energy integration, the plantwide interactions of the network are difficult to elucidate. Plantwide control has traditionally been implemented in a decentralized fashion, in which each subsystem is controlled independently and network interactions are treated as local subsystem disturbances [3,4]. It is well known, however, that when the inter-subsystem interactions are strong, decentralized control is unreliable [5].

Centralized control, in which all subsystems are controlled via a single agent, can account for the plantwide interactions. Indeed, increased computational power, faster optimization software, and algorithms designed specifically for large-scale plantwide control have made centralized control more practical [6,7]. Objections to centralized control are often not computational, however, but organizational. All subsystems rely upon the central agent, making plantwide control difficult to coordinate and maintain. These obstacles deter implementation of centralized control for large-scale plants.

As a middle ground between the decentralized and centralized strategies, distributed control preserves the topology and flexibility of decentralized control yet offers a nominal closed-loop stability guarantee. Stability is achieved by two features: the network interactions between subsystems are explicitly modeled and open-loop information, usually input trajectories, is exchanged between subsystem controllers. Distributed control strategies differ in the utilization of the open-loop information. In *noncooperative* distributed control, each subsystem controller anticipates the effect of network interactions only locally [8,9]. These strategies are described as noncooperative dynamic games [10], and the plantwide

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performance converges to the Nash equilibrium. If network interactions are strong, however, noncooperative control can destabilize the plant and performance may be, in these cases, poorer than decentralized control [11]. A more extensive and detailed comparison of cooperative and noncooperative approaches is provided in [12, pp.424–438].

Alternatively, *cooperative* distributed control improves performance by requiring each subsystem to consider the effect of local control actions on *all* subsystems in the network [13]. Each controller optimizes a plantwide objective function, e.g., the centralized controller objective. Distributed optimization algorithms are used to ensure a decrease in the plantwide objective at each intermediate iterate. Under cooperative control, plantwide performance converges to the Pareto optimum, providing centralized-like performance. Because the optimization may be terminated before convergence, however, cooperative control is a form of suboptimal control for the plantwide control problem. Hence, stability is deduced from suboptimal control theory [14].

Other recent work in large-scale control has focused on coordinating an underlying MPC structure. Aske et al. [15] develop a coordinating MPC that controls the plant variables with the greatest impact on plant performance, then allow the other decentralized controllers to react to the coordinator MPC. In a series of papers, Liu et al. [16] present a controller for networked, nonlinear subsystems [17,16]. A stabilizing decentralized control architecture and a control Lyapunov function are assumed to exist. The performance is improved via a coordinating controller that perturbs the network controller, taking into account the closed-loop response of the network. Cheng et al. [18] propose a distributed MPC that relies on a centralized dual optimization. This coordinator has the advantage that it can optimally handle coupling dynamics and constraints; it must wait for convergence of the plantwide problem, however, before it can provide an implementable input trajectory. Cooperative distributed MPC differs from these methods in that a coordinator is not necessary and suboptimal input trajectories may be used to stabilize the plant (see [19]).

In this paper, we state and prove the stability properties for cooperative distributed control under state and output feedback. In Section 2, we provide relevant theory for suboptimal control. Section 3 provides stability theory for cooperative control under state feedback. For ease of exposition, we introduce the theorems for the case of two controllers only. Section 4 extends these results to the output feedback case. The results are modified to handle coupled input constraints in Section 5. We then show how the theory extends to cover any finite number of controllers. We conclude with an example comparing the performance of cooperative control with other plantwide control strategies.

Notation. Given a vector $x \in \mathbb{R}^n$ the symbol $|x|$ denotes the Euclidean 2-norm; given a positive scalar r the symbol $\mathbb{B}_r$ denotes a closed ball of radius r centered at the origin, i.e. $\mathbb{B}_r = \{x \in \mathbb{R}^n, |x| \le r\}$. Given two integers, $l \le m$, we define the set $\mathbb{I}_{l:m} = \{l, l+1, \ldots, m-1, m\}$. The set of positive reals is denoted $\mathbb{R}_+$. The symbol $'$ indicates the matrix transpose.

2. Suboptimal model predictive control

Requiring distributed MPC strategies to converge is equivalent to implementing centralized MPC with the optimization distributed over many processors. Alternatively, we allow the subsystems to inject suboptimal inputs. This property increases the flexibility of distributed MPC, and the plantwide control strategy can be treated as a suboptimal MPC. In this section, we provide the definitions and theory of suboptimal MPC and draw upon these results in the sequel to establish stability of cooperative MPC.

We define the current state of the system as $x \in \mathbb{R}^n$, the trajectory of inputs $\mathbf{u} = \{u(0), u(1), \ldots, u(N-1)\} \in \mathbb{R}^{Nm}$, and the state and input at time k as $(x(k), u(k))$. For the latter, we often abbreviate the notation as (x, u). Denote the input constraints as $\mathbf{u} \in \mathbb{U}^N$ in which $\mathbb{U}$ is compact, convex, and contains the origin in its interior. Denote $\mathcal{X}_N$ as the set of all x for which there exists a feasible $\mathbf{u}$. Initialized with a feasible input trajectory $\tilde{\mathbf{u}}$, the controller performs p iterations of a feasible path algorithm and computes $\mathbf{u}$ such that some performance metric is improved. At each sample time, the first input in the (suboptimal) trajectory is applied, $u = u(0)$. The state is updated by the state evolution equation $x^+ = f(x, u)$, in which x^+ is the state at the next iterate.

For any initial state $x(0)$, we initialize the suboptimal MPC with a feasible input sequence $\tilde{\mathbf{u}}(0) = \mathbf{h}(x(0))$ with $\mathbf{h}(\cdot)$ continuous. For subsequent decision times, we denote $\tilde{\mathbf{u}}^+$ as the *warm start*, a feasible input sequence for x^+ used to initialize the suboptimal MPC algorithm. Here, we set $\tilde{\mathbf{u}}^+ = \{u(1), \ldots, u(N-1), 0\}$. This sequence is obtained by discarding the first input, shifting the rest of the sequence forward one step and setting the last input to zero.

We observe that the input sequence at termination $\mathbf{u}^+$ is a function of the state initial condition x^+ and of the warm start $\tilde{\mathbf{u}}^+$. Noting that x^+ and $\tilde{\mathbf{u}}^+$ are both functions of x and $\mathbf{u}$, the input sequence $\mathbf{u}^+$ can be expressed as a function of only $(x, \mathbf{u})$ by $\mathbf{u}^+ = g(x, \mathbf{u})$. We refer to the function g as the iterate update.

Given a system $x^+ = f(x)$, with equilibrium point at the origin $0 = f(0)$, denote $\phi(k, x(0))$ as the solution $x(k)$ given the initial state $x(0)$. We consider the following definition.

Definition 1 (*Exponential Stability on a Set* $\mathbb{X}$)**.** The origin is exponentially stable on the set $\mathbb{X}$ if for all $x(0) \in \mathbb{X}$, the solution $\phi(k, x(0)) \in \mathbb{X}$ and there exists $\alpha > 0$ and $0 < \gamma < 1$ such that

$$|\phi(k, x(0))| \le \alpha\, |x(0)|\, \gamma^k$$

for all $k \ge 0$.

The following lemma is an extension of [14, Theorem 1] for exponential stability.

Lemma 2 (*Exponential Stability of Suboptimal MPC*)**.** *Consider a system*

$$\begin{pmatrix} x^+ \\ \mathbf{u}^+ \end{pmatrix} = \begin{pmatrix} F(x, \mathbf{u}) \\ g(x, \mathbf{u}) \end{pmatrix} = \begin{pmatrix} f(x, u) \\ g(x, \mathbf{u}) \end{pmatrix} \qquad (x(0), \mathbf{u}(0)) \textit{ given} \tag{2.1}$$

with a steady-state solution $(0, 0) = (f(0, 0), g(0, 0))$. *Assume that the function* $V(\cdot) : \mathbb{R}^n \times \mathbb{R}^{Nm} \to \mathbb{R}_+$ *and input trajectory* $\mathbf{u}$ *satisfy*

$$a\, |(x, \mathbf{u})|^2 \le V(x, \mathbf{u}) \le b\, |(x, \mathbf{u})|^2 \tag{2.2a}$$

$$V(x^+, \mathbf{u}^+) - V(x, \mathbf{u}) \le -c\, |(x, u(0))|^2 \tag{2.2b}$$

$$|\mathbf{u}| \le d\, |x| \quad x \in \mathbb{B}_r \tag{2.2c}$$

in which $a, b, c, d, r > 0$. *If* $\mathcal{X}_N$ *is forward invariant for the system* $x^+ = f(x, u)$*, the origin is exponentially stable for all* $x(0) \in \mathcal{X}_N$.

Notice in the second inequality (2.2b), only the first input appears in the norm $|(x, u(0))|^2$. Note also that the last inequality applies only for x in a ball of radius r, which may be chosen arbitrarily small.

Proof of Lemma 2. First we establish that the origin of the extended system (2.1) is exponentially stable for all $(x(0), \mathbf{u}(0)) \in \mathcal{X}_N \times \mathbb{U}^N$. For $x \in \mathbb{B}_r$, we have $|\mathbf{u}| \le d\, |x|$. Consider the optimization

$$s = \max_{\mathbf{u} \in \mathbb{U}^N} |\mathbf{u}|\,.$$

The solution exists by the Weierstrass theorem since $\mathbb{U}^N$ is compact and by definition we have that $s > 0$. Then we have $|\mathbf{u}| \le (s/r)\, |x|$ for $x \notin \mathbb{B}_r$. Therefore, for all $x \in \mathcal{X}_N$, we have $|\mathbf{u}| \le \bar{d}\, |x|$ in which $\bar{d} = \max(d, s/r)$, and

$$|(x, \mathbf{u})| \le |x| + |\mathbf{u}| \le (1 + \bar{d})\, |x| \le (1 + \bar{d})\, |(x, u(0))|$$

which gives $|(x, u(0))| \geq \bar{c}\,|(x, \mathbf{u})|$ with $\bar{c} = 1/(1+\bar{d}) > 0$. Therefore the extended state $(x, \mathbf{u})$ satisfies

$$V(x^+, \mathbf{u}^+) - V(x, \mathbf{u}) \leq -\tilde{c}\,|(x, \mathbf{u})|^2 \quad (x, \mathbf{u}) \in \mathscr{X}_N \times \mathbb{U}^N \tag{2.3}$$

in which $\tilde{c} = c(\bar{c})^2$. Together with (2.2), (2.3) establishes that $V(\cdot)$ is a Lyapunov function of the extended state $(x, \mathbf{u})$ for all $x \in \mathscr{X}_N$ and $\mathbf{u} \in \mathbb{U}^N$. Hence for all $(x(0), \mathbf{u}(0)) \in \mathscr{X}_N \times \mathbb{U}^N$ and $k \geq 0$, we have

$$|(x(k), \mathbf{u}(k))| \leq \alpha\,|(x(0), \mathbf{u}(0))|\,\gamma^k$$

in which $\alpha > 0$ and $0 < \gamma < 1$. Notice that $\mathscr{X}_N \times \mathbb{U}^N$ is forward invariant for the extended system (2.1).

Finally, we remove the input sequence and establish that the origin is exponentially stable for the closed-loop system. We have for all $x(0) \in \mathscr{X}_N$ and $k \geq 0$

$$\begin{aligned}|\phi(k, x(0))| &= |x(k)| \leq |(x(k), \mathbf{u}(k))| \leq \alpha\,|(x(0), \mathbf{u}(0))|\,\gamma^k \\ &\leq \alpha(|x(0)| + |\mathbf{u}(0)|)\gamma^k \leq \alpha(1+\bar{d})\,|x(0)|\,\gamma^k \\ &\leq \bar{\alpha}\,|x(0)|\,\gamma^k\end{aligned}$$

in which $\bar{\alpha} = \alpha(1+\bar{d}) > 0$, and we have established exponential stability of the origin by observing that $\mathscr{X}_N$ is forward invariant for the closed-loop system $\phi(k, x(0))$. □

Remark 1. For Lemma 2, we use the fact that $\mathbb{U}$ is compact. For unbounded $\mathbb{U}$ exponential stability may instead be established by compactness of $\mathscr{X}_N$.

3. Cooperative model predictive control

We now show that cooperative MPC is a form of suboptimal MPC and establish stability. To simplify the exposition and proofs, in Sections 3–5 we assume that the plant consists of only two subsystems. We then establish in Section 6 that the results extend to any finite number of subsystems.

3.1. Definitions

3.1.1. Models

We assume for each subsystem i that there exist a collection of linear models denoting the effects of inputs of subsystem j on the states of subsystem i for all $(i, j) \in \mathbb{I}_{1:2} \times \mathbb{I}_{1:2}$

$$x_{ij}^+ = A_{ij}x_{ij} + B_{ij}u_j$$

in which $x_{ij} \in \mathbb{R}^{n_{ij}}$, $u_j \in \mathbb{R}^{m_j}$, $A_{ij} \in \mathbb{R}^{(n_{ij}\times n_{ij})}$, and $B_{ij} \in \mathbb{R}^{(n_{ij}\times m_j)}$. For a discussion of identification of this model choice, see [20]. The Appendix shows how these subsystem models are related to the centralized model. Considering subsystem 1, we collect the states to form

$$\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}^+ = \begin{bmatrix} A_{11} & \\ & A_{12} \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} + \begin{bmatrix} B_{11} \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ B_{12} \end{bmatrix} u_2$$

which denotes the model for subsystem 1. To simplify the notation, we define the equivalent model

$$x_1^+ = A_1x_1 + \bar{B}_{11}u_1 + \bar{B}_{12}u_2$$

for which

$$x_1 = \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} \qquad A_1 = \begin{bmatrix} A_{11} & \\ & A_{12} \end{bmatrix} \quad \bar{B}_{11} = \begin{bmatrix} B_{11} \\ 0 \end{bmatrix} \qquad \bar{B}_{12} = \begin{bmatrix} 0 \\ B_{12} \end{bmatrix}$$

in which $x_1 \in \mathbb{R}^{n_1}$, $A_1 \in \mathbb{R}^{(n_1\times n_1)}$, and $\bar{B}_{1j} \in \mathbb{R}^{(n_1\times m_j)}$ with $n_1 = n_{11} + n_{12}$. Forming a similar model for subsystem 2, the plantwide model is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^+ = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_{11} \\ \bar{B}_{21} \end{bmatrix} u_1 + \begin{bmatrix} \bar{B}_{12} \\ \bar{B}_{22} \end{bmatrix} u_2.$$

We further simplify the plantwide model notation to

$$x^+ = Ax + B_1u_1 + B_2u_2$$

for which

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad A = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix} \qquad B_1 = \begin{bmatrix} \bar{B}_{11} \\ \bar{B}_{21} \end{bmatrix} \qquad B_2 = \begin{bmatrix} \bar{B}_{12} \\ \bar{B}_{22} \end{bmatrix}.$$

3.1.2. Objective functions

Consider subsystem 1, for which we define the quadratic stage cost and terminal penalty, respectively

$$\ell_1(x_1, u_1) = \frac{1}{2}(x_1'Q_1x_1 + u_1'R_1u_1) \tag{3.1a}$$

$$V_{1f}(x_1) = \frac{1}{2}x_1'P_{1f}x_1 \tag{3.1b}$$

in which $Q_1 \in \mathbb{R}^{(n_1\times n_1)}$, $R_1 \in \mathbb{R}^{(m_1\times m_1)}$, and $P_{1f} \in \mathbb{R}^{(n_1\times n_1)}$. We define the objective function for subsystem 1

$$V_1\big(x_1(0), \mathbf{u}_1, \mathbf{u}_2\big) = \sum_{k=0}^{N-1} \ell_1\big(x_1(k), u_1(k)\big) + V_{1f}\big(x_1(N)\big).$$

Notice V_1 is implicitly a function of both $\mathbf{u}_1$ and $\mathbf{u}_2$ because x_1 is a function of both u_1 and u_2. For subsystem 2, we similarly define an objective function V_2. We define the plantwide objective function

$$\begin{aligned}V\big(x_1(0), x_2(0), \mathbf{u}_1, \mathbf{u}_2\big) &= \rho_1V_1\big(x_1(0), \mathbf{u}_1, \mathbf{u}_2\big) \\ &\quad + \rho_2V_2\big(x_2(0), \mathbf{u}_1, \mathbf{u}_2\big)\end{aligned}$$

in which $\rho_1, \rho_2 > 0$ are relative weights. For notational simplicity, we write $V(x, \mathbf{u})$ for the plant objective.

3.1.3. Constraints

We require that the inputs satisfy

$$u_1(k) \in \mathbb{U}_1 \qquad u_2(k) \in \mathbb{U}_2 \quad k \in \mathbb{I}_{0:N-1}$$

in which $\mathbb{U}_1$ and $\mathbb{U}_2$ are compact and convex such that 0 is in the interior of $\mathbb{U}_i\,\forall i \in \mathbb{I}_{1:2}$.

Remark 2. The constraints are termed *uncoupled* because the feasible region of $\mathbf{u}_1$ is not affected by $\mathbf{u}_2$, and vice versa.

3.1.4. Assumptions

For every $i \in \mathbb{I}_{1:2}$, let $\underline{A}_i = \mathrm{diag}(A_{1i}, A_{2i})$ and $\underline{B}_i = \begin{bmatrix} B_{1i} \\ B_{2i} \end{bmatrix}$. The following assumptions are used to establish stability.

Assumption 3. For all $i \in \mathbb{I}_{1:2}$

(a) The systems $(\underline{A}_i, \underline{B}_i)$ are stabilizable.
(b) The input penalties $R_i > 0$.
(c) The state penalties $Q_i \geq 0$.
(d) The systems (A_i, Q_i) are detectable.
(e) $N \geq \max_{i\in\mathbb{I}_{1:2}}(\underline{n}_i^u)$, in which $\underline{n}_i^u$ is the number of unstable modes of $\underline{A}_i$, i.e., the number of $\lambda \in \mathrm{eig}(\underline{A}_i)$ such that $|\lambda| \geq 1$.

The Assumption 3(e) is required so that the horizon N is sufficiently large to zero the unstable modes.

3.1.5. Unstable modes

For an unstable plant, we constrain the unstable modes to be zero at the end of the horizon to maintain closed-loop stability. To construct this constraint, consider the real Schur decomposition of A_{ij} for each $(i, j) \in \mathbb{I}_{1:2} \times \mathbb{I}_{1:2}$

$$A_{ij} = \begin{bmatrix} S_{ij}^s & S_{ij}^u \end{bmatrix}\begin{bmatrix} A_{ij}^s & - \\ & A_{ij}^u \end{bmatrix}\begin{bmatrix} S_{ij}^{s\,\prime} \\ S_{ij}^{u\,\prime} \end{bmatrix} \tag{3.2}$$

in which A_{ij}^s is stable and A_{ij}^u has all unstable eigenvalues.

3.1.6. Terminal penalty

Given the definition of the Schur decomposition (3.2), we define the matrices

$$S_i^s = \mathrm{diag}(S_{i1}^s, S_{i2}^s) \qquad A_i^s = \mathrm{diag}(A_{i1}^s, A_{i2}^s) \quad \forall i \in \mathbb{I}_{1:2} \tag{3.3a}$$

$$S_i^u = \mathrm{diag}(S_{i1}^u, S_{i2}^u) \qquad A_i^u = \mathrm{diag}(A_{i1}^u, A_{i2}^u) \quad \forall i \in \mathbb{I}_{1:2}. \tag{3.3b}$$

Lemma 4. *The matrices (3.3) satisfy the Schur decompositions*

$$A_i = \begin{bmatrix} S_i^s & S_i^u \end{bmatrix} \begin{bmatrix} A_i^s & - \\ & A_i^u \end{bmatrix} \begin{bmatrix} S_i^{s\prime} \\ S_i^{u\prime} \end{bmatrix} \quad \forall i \in \mathbb{I}_{1:2}.$$

Let Σ_1 and Σ_2 denote the solution of the Lyapunov equations

$$A_1^{s\prime} \Sigma_1 A_1^s - \Sigma_1 = -S_1^{s\prime} Q_1 S_1^s \qquad A_2^{s\prime} \Sigma_2 A_2^s - \Sigma_2 = -S_2^{s\prime} Q_2 S_2^s. \tag{3.4}$$

We then choose the terminal penalty for each subsystem to be the cost to go under zero control, such that

$$P_{1f} = S_1^s \Sigma_1 S_1^{s\prime} \qquad P_{2f} = S_2^s \Sigma_2 S_2^{s\prime}. \tag{3.5}$$

3.1.7. Cooperative model predictive control algorithm

Let $\boldsymbol{\upsilon}^0$ be the initial condition for the cooperative MPC algorithm (see Section 3.2 for the discussion of initialization). At each iterate $p \geq 0$, the following optimization problem is solved for subsystem i, $i \in \mathbb{I}_{1:2}$

$$\min_{\boldsymbol{\upsilon}_i} V(x_1(0), x_2(0), \boldsymbol{\upsilon}_1, \boldsymbol{\upsilon}_2) \tag{3.6a}$$

subject to

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^+ = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_{11} \\ \bar{B}_{21} \end{bmatrix} \upsilon_1 + \begin{bmatrix} \bar{B}_{12} \\ \bar{B}_{22} \end{bmatrix} \upsilon_2 \tag{3.6b}$$

$$\boldsymbol{\upsilon}_i \in \mathbb{U}_i^N \tag{3.6c}$$

$$S_{ji}^{u\prime} x_{ji}(N) = 0 \quad j \in \mathbb{I}_{1:2} \tag{3.6d}$$

$$|\boldsymbol{\upsilon}_i| \leq d_i \sum_{j \in \mathbb{I}_{1:2}} |x_{ji}(0)| \quad \text{if } x_{ji}(0) \in \mathbb{B}_r \ \forall j \in \mathbb{I}_{1:2} \tag{3.6e}$$

$$\boldsymbol{\upsilon}_j = \boldsymbol{\upsilon}_j^p \ j \in \mathbb{I}_{1:2} \setminus i \tag{3.6f}$$

in which we include the hard input constraints, the stabilizing constraint on the unstable modes, and the Lyapunov stability constraint. We denote the solutions to these problems as

$$\boldsymbol{\upsilon}_1^*(x_1(0), x_2(0), \boldsymbol{\upsilon}_2^p), \qquad \boldsymbol{\upsilon}_2^*(x_1(0), x_2(0), \boldsymbol{\upsilon}_1^p).$$

Given the prior, feasible iterate $(\boldsymbol{\upsilon}_1^p, \boldsymbol{\upsilon}_2^p)$, the next iterate is defined to be

$$(\boldsymbol{\upsilon}_1^{p+1}, \boldsymbol{\upsilon}_2^{p+1}) = w_1 \left(\boldsymbol{\upsilon}_1^*(\boldsymbol{\upsilon}_2^p), \boldsymbol{\upsilon}_2^p \right) + w_2 \left(\boldsymbol{\upsilon}_1^p, \boldsymbol{\upsilon}_2^*(\boldsymbol{\upsilon}_1^p) \right) \tag{3.7}$$

$$w_1 + w_2 = 1, \qquad w_1, w_2 > 0$$

for which we omit the state dependence of $\boldsymbol{\upsilon}_1^*$ and $\boldsymbol{\upsilon}_2^*$ to reduce notation. This distributed optimization is of the Gauss–Jacobi type (see [21], pp. 219–223). At the last iterate $\bar{p}$, we set $\mathbf{u} \leftarrow (\boldsymbol{\upsilon}_1^{\bar{p}}, \boldsymbol{\upsilon}_2^{\bar{p}})$ and inject $u(0)$ into the plant.

The following properties follow immediately.

Lemma 5 (*Feasibility*)**.** *Given a feasible initial guess, the iterates satisfy*

$$(\boldsymbol{\upsilon}_1^p, \boldsymbol{\upsilon}_2^p) \in \mathbb{U}_1^N \times \mathbb{U}_2^N$$

for all $p \geq 1$.

Lemma 6 (*Convergence*)**.** *The cost* $V(x(0), \boldsymbol{\upsilon}^p)$ *is nonincreasing for each iterate* p *and converges as* $p \to \infty$.

Lemma 7 (*Optimality*)**.** *As* $p \to \infty$ *the cost* $V(x(0), \boldsymbol{\upsilon}^p)$ *converges to the optimal value* $V^0(x(0))$*, and the iterates* $(\boldsymbol{\upsilon}_1^p, \boldsymbol{\upsilon}_2^p)$ *converge to* $(\mathbf{u}_1^0, \mathbf{u}_2^0)$ *in which* $\mathbf{u}^0 = (\mathbf{u}_1^0, \mathbf{u}_2^0)$ *is the Pareto (centralized) optimal solution.*

The proofs are in the Appendix.

Remark 3. This paper presents the distributed optimization algorithm with subproblem (3.6) and iterate update (3.7) so that the Lemmas 5–7 are satisfied. This choice is nonunique and other optimization methods may exist satisfying these properties.

3.2. Stability of cooperative model predictive control

We define the steerable set $\mathcal{X}_N$ as the set of all x such that there exists a $\mathbf{u} \in \mathbb{U}^N$ satisfying (3.6d).

Assumption 8. Given $r > 0$, for all $i \in \mathbb{I}_{1:2}$, d_i is chosen large enough such that there exists a $\mathbf{u}_i \in \mathbb{U}^N$ satisfying $|\mathbf{u}_i| \leq d_i \sum_{j \in \mathbb{I}_{1:2}} |x_{ij}|$ and (3.6d) for all $x_{ij} \in \mathbb{B}_r \forall j \in \mathbb{I}_{1:2}$.

Remark 4. Given Assumption 8, $\mathcal{X}_N$ is forward invariant.

We now establish stability of the closed-loop system by treating cooperative MPC as a form of suboptimal MPC. We define the warm start for each subsystem as

$$\tilde{\mathbf{u}}_1^+ = \{u_1(1), u_1(2), \ldots, u_1(N-1), 0\}$$

$$\tilde{\mathbf{u}}_2^+ = \{u_2(1), u_2(2), \ldots, u_2(N-1), 0\}.$$

The warm start $\tilde{\mathbf{u}}_i^+$ is used as the initial condition for the cooperative MPC problem in each subsystem i. We define the functions g_1^p and g_2^p as the outcome of applying the cooperative control iteration (3.7) p times

$$\mathbf{u}_1^+ = g_1^p(x_1, x_2, \mathbf{u}_1, \mathbf{u}_2) \qquad \mathbf{u}_2^+ = g_2^p(x_1, x_2, \mathbf{u}_1, \mathbf{u}_2).$$

The system evolution is then given by

$$\begin{pmatrix} x_1^+ \\ x_2^+ \\ \mathbf{u}_1^+ \\ \mathbf{u}_2^+ \end{pmatrix} = \begin{pmatrix} A_1 x_1 + \bar{B}_{11} u_1 + \bar{B}_{12} u_2 \\ A_2 x_2 + \bar{B}_{21} u_1 + \bar{B}_{22} u_2 \\ g_1^p(x_1, x_2, \mathbf{u}_1, \mathbf{u}_2) \\ g_2^p(x_1, x_2, \mathbf{u}_1, \mathbf{u}_2) \end{pmatrix}$$

which we simplify to

$$\begin{pmatrix} x^+ \\ \mathbf{u}^+ \end{pmatrix} = \begin{pmatrix} Ax + B_1 u_1 + B_2 u_2 \\ g^p(x, \mathbf{u}) \end{pmatrix}.$$

Theorem 9 (*Exponential Stability*)**.** *Given Assumptions 3 and 8, the origin* $(x = 0)$ *of the closed-loop system* $x^+ = Ax + B_1 u_1 + B_2 u_2$ *is exponentially stable on the set* $\mathcal{X}_N$.

Proof. By eliminating the states in $\ell_i(\cdot)$, we can write V in the form $V(x, \mathbf{u}) = (1/2)x'\mathcal{Q}x + (1/2)\mathbf{u}'\mathcal{R}\mathbf{u} + x'\mathcal{S}\mathbf{u}$. Defining $\mathcal{H} = \begin{bmatrix} \mathcal{Q} & \mathcal{S} \\ \mathcal{S}' & \mathcal{R} \end{bmatrix} > 0$, $V(\cdot)$ satisfies (2.2a) by choosing $a = (1/2) \min_i(\lambda_i(\mathcal{H}))$ and $b = (1/2) \max_i(\lambda_i(\mathcal{H}))$. Next we show that $V(\cdot)$ satisfies (2.2b). Using the warm start at the next sample time, we have the following cost

$$\begin{aligned} V(x^+, \tilde{\mathbf{u}}^+) &= V(x, \mathbf{u}) - \frac{1}{2}\rho_1 \ell_1(x_1, u_1) - \frac{1}{2}\rho_2 \ell_2(x_2, u_2) \\ &\quad + \frac{1}{2}\rho_1 x_1(N)' \Big(A_1' P_{1f} A_1 - P_{1f} + Q_1 \Big) x_1(N) \\ &\quad + \frac{1}{2}\rho_2 x_2(N)' \Big(A_2' P_{2f} A_2 - P_{2f} + Q_2 \Big) x_2(N). \end{aligned} \tag{3.8}$$

Using the Schur decomposition defined in Lemma 4, and the constraints (3.6d) and (3.5), the last two terms of (3.8) can be written as

$$\frac{1}{2}\rho_1 x_1(N)' S_1^s \left(A_1^{s\prime} \Sigma_1 A_1^s - \Sigma_1 + S_1^{s\prime} Q_1 S_1^s \right) S_1^{s\prime} x_1(N)$$
$$+ \frac{1}{2}\rho_2 x_2(N)' S_2^s \left(A_2^{s\prime} \Sigma_2 A_2^s - \Sigma_2 + S_2^{s\prime} Q_2 S_2^s \right) S_2^{s\prime} x_2(N) = 0.$$

These terms are zero because of (3.4). Using this result and applying the iteration of the controllers gives

$$V(x^+, \mathbf{u}^+) \le V(x, \mathbf{u}) - \frac{1}{2}\rho_1 \ell_1(x_1, u_1) - \frac{1}{2}\rho_2 \ell_2(x_2, u_2).$$

Because ℓ_i is quadratic in both arguments, there exists a $c > 0$ such that

$$V(x^+, \mathbf{u}^+) - V(x, \mathbf{u}) \le -c\,|(x, u)|^2 .$$

The Lyapunov stability constraint (3.6e) for $x_{11}, x_{12}, x_{21}, x_{22} \in \mathbb{B}_r$ implies for $(x_1, x_2) \in \mathbb{B}_r$ that $|(\mathbf{u}_1, \mathbf{u}_2)| \le 2\hat{d}\,|(x_1, x_2)|$ in which $\hat{d} = \max(d_1, d_2)$, satisfying (2.2c). Therefore the closed-loop system satisfies Lemma 2. Hence the closed-loop system is exponentially stable. □

4. Output feedback

We now consider the stability of the closed-loop system with estimator error.

4.1. Models

For all $(i, j) \in \mathbb{I}_{1:2} \times \mathbb{I}_{1:2}$

$$x_{ij}^+ = A_{ij} x_{ij} + B_{ij} u_j \qquad (4.1a)$$

$$y_i = \sum_{j \in \mathbb{I}_{1:2}} C_{ij} x_{ij} \qquad (4.1b)$$

in which $y_i \in \mathbb{R}^{p_i}$ is the output of subsystem i and $C_{ij} \in \mathbb{R}^{(p_i \times n_{ij})}$. Consider subsystem 1. As above, we collect the states to form $y_1 = [C_{11} \quad C_{12}]\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$ and use the simplified notation $y_1 = C_1 x_1$ to form the output model for subsystem 1.

Assumption 10. For all $i \in \mathbb{I}_{1:2}$, (A_i, C_i) is detectable.

4.2. Estimator

We construct a decentralized estimator. Consider subsystem 1, for which the local measurement y_1 and both inputs u_1 and u_2 are available, but x_1 must be estimated. The estimate satisfies

$$\hat{x}_1^+ = A_1 \hat{x}_1 + \bar{B}_{11} u_1 + \bar{B}_{12} u_2 + L_1 (y_1 - C_1 \hat{x}_1)$$

in which $\hat{x}_1$ is the estimate of x_1 and L_1 is the Kalman filter gain. Defining the estimate error as $e_1 = x_1 - \hat{x}_1$ we have $e_1^+ = (A_1 - L_1 C_1) e_1$. By Assumptions 3 and 10 there exists an L_1 such that $(A_1 - L_1 C_1)$ is stable and therefore the estimator for subsystem 1 is stable. Defining e_2 similarly, the estimate error for the plant evolves

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}^+ = \begin{bmatrix} A_{L1} & \\ & A_{L2} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

in which $A_{Li} = A_i - L_i C_i$. We collect the estimate error of each subsystem together and write $e^+ = A_L e$.

4.3. Reinitialization

We define the reinitialization step required to recover feasibility of the warm start for the perturbed state terminal constraint. For each $i \in \mathbb{I}_{1:2}$, define

$$\mathbf{h}_i^+(e) = \arg\min_{\mathbf{u}_i} \left\{ \left| \mathbf{u}_i - \tilde{\mathbf{u}}_i^+ \right|^2_{\mathscr{R}_i} \middle| \begin{array}{c} F_{ji}(\mathbf{u}_i - \tilde{\mathbf{u}}_i^+) = f_{ji} e_j \ \forall j \in \mathbb{I}_{1:2} \\ \mathbf{u}_i \in \mathbb{U}^N \end{array} \right\}$$

in which $\mathscr{R}_i = \operatorname{diag}(R_i)$, $F_{ji} = S_{ji}^{u\prime} \mathscr{C}_{ji}$, $f_{ji} = -S_{ji}^{u\prime} A_{ji}^N L_{ji}$, and $\mathscr{C}_{ji} = [B_{ji} \ A_{ji} B_{ji} \ \cdots \ A_{ji}^{N-1} B_{ji}]$ for all $i, j \in \mathbb{I}_{1:2}$. We use $\mathbf{h}_i^+(e)$ as the initial condition for the control optimization (3.6) for all $i \in \mathbb{I}_{1:2}$.

Proposition 11. *The reinitialization $\mathbf{h}^+(\cdot) = (\mathbf{h}_1^+(\cdot), \mathbf{h}_2^+(\cdot))$ is Lipschitz continuous on bounded sets.*

Proof. The proof follows from Prop. 7.13 [12, p.499]. □

4.4. Stability with estimate error

We consider the stability properties of the extended closed-loop system

$$\begin{pmatrix} \hat{x} \\ \mathbf{u} \\ e \end{pmatrix}^+ = \begin{pmatrix} F(\hat{x}, \mathbf{u}) + Le \\ g^p(\hat{x}, \mathbf{u}, e) \\ A_L e \end{pmatrix} \qquad (4.2)$$

in which $F(\hat{x}, \mathbf{u}) = A\hat{x} + B_1 u_1 + B_2 u_2$ and $L = \operatorname{diag}(L_1 C_1, L_2 C_2)$. The function g^p includes the reinitialization step. Because A_L is stable there exists a Lyapunov function $J(\cdot)$ with the following properties

$$\bar{a}\,|e|^\sigma \le J(e) \le \bar{b}\,|e|^\sigma$$
$$J(e^+) - J(e) \le -\bar{c}\,|e|^\sigma$$

in which $\sigma > 0$, $\bar{a}, \bar{b} > 0$, and the constant $\bar{c} > 0$ can be chosen as large as desired by scaling $J(\cdot)$. For the remainder of this section, we choose $\sigma = 1$ in order to match the Lipschitz continuity of the plantwide objective function $V(\cdot)$. From the nominal properties of cooperative MPC, the origin of the nominal closed-loop system $x^+ = Ax + B_1 u_1 + B_2 u_2$ is exponentially stable on $\mathscr{X}_N$ if the suboptimal input trajectory $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ is computed using the actual state x, and the cost function $V(x, \mathbf{u})$ satisfies (2.2). We require the following feasibility assumption.

Assumption 12. The set $\mathscr{X}_N$ is compact, and there exist two sets $\hat{\mathscr{X}}_N$ and $\mathscr{E}$ both containing the origin such that the following conditions hold: (i) $\hat{\mathscr{X}}_N \oplus \mathscr{E} \subseteq \mathscr{X}_N$, where $\oplus$ indicates the Minkowski sum; (ii) for each $\hat{x}(0) \in \hat{\mathscr{X}}_N$ and $\hat{e}(0) \in \mathscr{E}$, the solution of the extended closed-loop system (4.2) satisfies $\hat{x}(k) \in \mathscr{X}_N$ for all $k \ge 0$.

Consider the sum of the two Lyapunov functions

$$W(\hat{x}, \mathbf{u}, e) = V(\hat{x}, \mathbf{u}) + J(e).$$

We next show that $W(\cdot)$ is a Lyapunov function for the perturbed system and establish exponential stability of the extended state origin $(\hat{x}, e) = (0, 0)$. From the definition of $W(\cdot)$ we have

$$a\left|(\hat{x}, \mathbf{u})\right|^2 + \bar{a}\,|e| \le W(\hat{x}, \mathbf{u}, e) \le b\left|(\hat{x}, \mathbf{u})\right|^2 + \bar{b}\,|e|$$
$$\Longrightarrow \tilde{a}\left(\left|(\hat{x}, \mathbf{u})\right|^2 + |e|\right) \le W(\hat{x}, \mathbf{u}, e) \le \tilde{b}(\left|(\hat{x}, \mathbf{u})\right|^2 + |e|) \qquad (4.3)$$

in which $\tilde{a} = \min(a, \bar{a}) > 0$ and $\tilde{b} = \max(b, \bar{b}) > 0$. Next we compute the cost change

$$W(\hat{x}^+, \mathbf{u}^+, e^+) - W(\hat{x}, \mathbf{u}, e)$$
$$= V(\hat{x}^+, \mathbf{u}^+) - V(\hat{x}, \mathbf{u}) + J(e^+) - J(e).$$

The Lyapunov function V is quadratic in $(\hat{x}, \mathbf{u})$ and, hence, Lipschitz continuous on bounded sets. By Proposition 11

$$\left|V(F(\hat{x}, \mathbf{u}) + Le, \mathbf{h}^+(e)) - V(F(\hat{x}, \mathbf{u}) + Le, \tilde{\mathbf{u}}^+)\right| \le L_h L_{V_u}\,|e|$$
$$\left|V(F(\hat{x}, \mathbf{u}) + Le, \tilde{\mathbf{u}}^+) - V(F(\hat{x}, \mathbf{u}), \tilde{\mathbf{u}}^+)\right| \le L_{V_x}\,|Le|$$

in which L_h, L_{V_u}, and L_{V_x} are Lipschitz constants for $\mathbf{h}^+$ and the first and second arguments of V, respectively. Combining the above inequalities

$$\left|V(F(\hat{x}, \mathbf{u}) + Le, \mathbf{h}^+(e)) - V(F(\hat{x}, \mathbf{u}), \tilde{\mathbf{u}}^+)\right| \le \bar{L}_V\,|e|$$

in which $\bar{L}_V = L_h L_{V_u} + L_{V_x} |L|$. Using the system evolution we have

$$V(\hat{x}^+, \mathbf{h}^+(e)) \le V(F(\hat{x}, \mathbf{u}), \tilde{\mathbf{u}}^+) + \bar{L}_V |e|$$

and by Lemma 6

$$V(\hat{x}^+, \mathbf{u}^+) \le V(F(\hat{x}, \mathbf{u}), \tilde{\mathbf{u}}^+) + \bar{L}_V |e| .$$

Subtracting $V(\hat{x}, \mathbf{u})$ from both sides and noting that $\tilde{\mathbf{u}}^+$ is a stabilizing input sequence for $e = 0$ gives

$$V(\hat{x}^+, \mathbf{u}^+) - V(\hat{x}, \mathbf{u}) \le -c \left|(\hat{x}, u(0))\right|^2 + \bar{L}_V |e|$$

$$\begin{aligned} W(\hat{x}^+, \mathbf{u}^+, e^+) - W(\hat{x}, \mathbf{u}, e) &\le -c \left|(\hat{x}, u(0))\right|^2 + \bar{L}_V |e| - \bar{c} |e| \\ &\le -c \left|(\hat{x}, u(0))\right|^2 - (\bar{c} - \bar{L}_V) |e| \end{aligned}$$

$$W(\hat{x}^+, \mathbf{u}^+, e^+) - W(\hat{x}, \mathbf{u}, e) \le -\tilde{c}(\left|(\hat{x}, u(0))\right|^2 + |e|) \tag{4.4}$$

in which we choose $\bar{c} > \bar{L}_V$ and $\tilde{c} = \min(c, \bar{c} - \bar{L}_V) > 0$. This choice is possible because $\bar{c}$ can be chosen arbitrarily large. Notice this step is what motivated the choice of $\sigma = 1$. Lastly, we require the constraint

$$|\mathbf{u}| \le d \left|\hat{x}\right|, \quad \hat{x} \in \mathbb{B}_r. \tag{4.5}$$

Theorem 13 (*Exponential Stability of Perturbed System*). *Given Assumptions 3, 10 and 12, for each $\hat{x}(0) \in \mathcal{X}_N$ and $e(0) \in \mathcal{E}$, there exist constants $\alpha > 0$ and $0 < \gamma < 1$, such that the solution of the perturbed system (4.2) satisfies, for all $k \ge 0$*

$$|(\hat{x}(k), e(k))| \le \alpha |(\hat{x}(0), e(0))| \gamma^k. \tag{4.6}$$

Proof. Using the same arguments as for Lemma 2, we write:

$$W(\hat{x}^+, \mathbf{u}^+, e^+) - W(\hat{x}, \mathbf{u}, e) \le -\hat{c}(\left|(\hat{x}, \mathbf{u})\right|^2 + |e|) \tag{4.7}$$

in which $\hat{c} \ge \tilde{c} > 0$. Therefore $W(\cdot)$ is a Lyapunov function for the extended state $(\hat{x}, \mathbf{u}, e)$ with mixed norm powers. The standard exponential stability argument can be extended for the mixed norm power case to show that the origin of the extended closed-loop system (4.2) is exponentially stable [12, p.420]. Hence, for all $k \ge 0$

$$\left|(\hat{x}(k), \mathbf{u}(k), e(k))\right| \le \tilde{\alpha} \left|(\hat{x}(0), \mathbf{u}(0), e(0))\right| \gamma^k$$

in which $\tilde{\alpha} > 0$ and $0 < \gamma < 1$. Notice that Assumption 12 implies that $\mathbf{u}(k)$ exists for all $k \ge 0$ because $\hat{x}(k) \in \mathcal{X}_N$.

We have, using the same arguments used in Lemma 2

$$\begin{aligned} \left|(\hat{x}(k), e(k))\right| &\le \left|(\hat{x}(k), \mathbf{u}(k), e(k))\right| \le \tilde{\alpha} \left|(\hat{x}(0), \mathbf{u}(0), e(0))\right| \gamma^k \\ &\le \alpha \left|(\hat{x}(0), e(0))\right| \gamma^k \end{aligned}$$

in which $\alpha = \tilde{\alpha}(1 + \bar{d}) > 0$. □

Corollary 14. *Under the assumptions of Theorem 13, for each $x(0)$ and $\hat{x}(0)$ such that $e(0) = x(0) - \hat{x}(0) \in \mathcal{E}$ and $\hat{x}(0) \in \mathcal{X}_N$, the solution of the closed-loop state $x(k) = \hat{x}(k) + e(k)$ satisfies:*

$$|x(k)| \le \bar{\alpha} |x(0)| \gamma^k \tag{4.8}$$

for some $\bar{\alpha} > 0$ and $0 < \gamma < 1$.

5. Coupled constraints

In Remark 2, we commented that the constraint assumptions imply uncoupled constraints, because each input is constrained by a separate feasible region so that the full feasible space is defined $(\mathbf{u}_1, \mathbf{u}_2) \in \mathbb{U}^N = \mathbb{U}_1^N \times \mathbb{U}_2^N$. This assumption, however, is not always practical. Consider two subsystems sharing a scarce resource for which we control the distribution. There then exists an availability constraint spanning the subsystems. This constraint is *coupled* because each local resource constraint depends upon the amount requested by the other subsystem.

Remark 5. For plants with coupled constraints, implementing MPC problem (3.6) gives exponentially stable, yet suboptimal, feedback.

In this section, we relax the assumption so that $(\mathbf{u}_1, \mathbf{u}_2) \in \mathbb{U}^N$ for any $\mathbb{U}$ compact, convex and containing the origin in its interior. Consider the decomposition of the inputs $\mathbf{u} = (\mathbf{u}_{U1}, \mathbf{u}_{U2}, \mathbf{u}_C)$ such that there exists a $\mathbb{U}_{U1}$,$\mathbb{U}_{U2}$, and $\mathbb{U}_C$ for which

$$\mathbb{U} = \mathbb{U}_{U1} \times \mathbb{U}_{U2} \times \mathbb{U}_C$$

and

$$\mathbf{u}_{U1} \in \mathbb{U}_{U1}^N, \qquad \mathbf{u}_{U2} \in \mathbb{U}_{U2}^N, \qquad \mathbf{u}_C \in \mathbb{U}_C^N$$

for which $\mathbb{U}_{U1}$, $\mathbb{U}_{U2}$, and $\mathbb{U}_C$ are compact and convex. We denote $\mathbf{u}_{Ui}$ the uncoupled inputs for subsystem i, $i \in \mathbb{I}_{1:2}$, and $\mathbf{u}_C$ the coupled inputs.

Remark 6. $\mathbb{U}_{U1}$,$\mathbb{U}_{U2}$, or $\mathbb{U}_C$ may be empty, and therefore such a decomposition always exists.

We modify the cooperative MPC problem (3.6) for the above decomposition. Define the augmented inputs $(\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2)$

$$\hat{\mathbf{u}}_1 = \begin{bmatrix} \mathbf{u}_{U1} \\ \mathbf{u}_C \end{bmatrix} \qquad \hat{\mathbf{u}}_2 = \begin{bmatrix} \mathbf{u}_{U2} \\ \mathbf{u}_C \end{bmatrix}.$$

The implemented inputs are

$$\mathbf{u}_1 = \hat{E}_1 \hat{\mathbf{u}}_1 \qquad \mathbf{u}_2 = \hat{E}_2 \hat{\mathbf{u}}_2, \quad \hat{E}_1 = \begin{bmatrix} I & \\ & I_1 \end{bmatrix} \quad \hat{E}_2 = \begin{bmatrix} I & \\ & I_2 \end{bmatrix}$$

in which (I_1, I_2) are diagonal matrices with either 0 or 1 diagonal entries and satisfy $I_1 + I_2 = I$. For simplicity, we summarize the previous relations as $\mathbf{u} = \hat{E}\hat{\mathbf{u}}$ with $\hat{E} = \mathrm{diag}(\hat{E}_1, \hat{E}_2)$. The objective function is

$$\hat{V}(x_1(0), x_2(0), \hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2) = V(x_1(0), x_2(0), \hat{E}_1 \hat{\mathbf{u}}_1, \hat{E}_2 \hat{\mathbf{u}}_2). \tag{5.1}$$

We solve the augmented cooperative MPC problem for $i \in \mathbb{I}_{1:2}$

$$\min_{\hat{\mathbf{v}}_i} \hat{V}(x_1(0), x_2(0), \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2) \tag{5.2a}$$

subject to

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^+ = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_{11} \\ \bar{B}_{21} \end{bmatrix} \hat{E}_1 \hat{\upsilon}_1 + \begin{bmatrix} \bar{B}_{12} \\ \bar{B}_{22} \end{bmatrix} \hat{E}_2 \hat{\upsilon}_2 \tag{5.2b}$$

$$\hat{\mathbf{v}}_i \in \mathbb{U}_{Ui}^N \times \mathbb{U}_C^N \tag{5.2c}$$

$$S_{ji}^{u\prime} x_{ji}(N) = 0 \quad j \in \mathbb{I}_{1:2} \tag{5.2d}$$

$$\left|\hat{\mathbf{v}}_i\right| \le d_i |x_i(0)| \quad \text{if } x_i(0) \in \mathbb{B}_r \tag{5.2e}$$

$$\hat{\mathbf{v}}_j = \hat{\mathbf{v}}_j^p \quad j \in \mathbb{I}_{1:2} \setminus i. \tag{5.2f}$$

The update (3.7) is used to determine the next iterate.

Lemma 15. *As $p \to \infty$ the cost $\hat{V}(x(0), \hat{\mathbf{v}}^p)$ converges to the optimal value $V^0(x(0))$, and the iterates $(\hat{E}_1 \hat{\mathbf{v}}_1^p, \hat{E}_2 \hat{\mathbf{v}}_2^p)$ converge to the Pareto optimal centralized solution $\mathbf{u}^0 = (\mathbf{u}_1^0, \mathbf{u}_2^0)$.*

Therefore, problem (5.2) gives optimal feedback and may be used for plants with coupled constraints.

6. *M* Subsystems

In this section, we show that the stability theory of cooperative control extends to any finite number of subsystems. For $M > 0$

subsystems, the plantwide variables are defined

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_M \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_M \end{bmatrix} \qquad B_i = \begin{bmatrix} \bar{B}_{1i} \\ \bar{B}_{2i} \\ \vdots \\ \bar{B}_{Mi} \end{bmatrix} \qquad \forall i \in \mathbb{I}_{1:M}$$

$$V(x, \mathbf{u}) = \sum_{i \in \mathbb{I}_{1:M}} \rho_i V_i(x_i, \mathbf{u}_i) \qquad A = \mathrm{diag}(A_1, \ldots, A_M).$$

Each subsystem solves the optimization

$$\min_{\boldsymbol{\upsilon}_i} V(x(0), \boldsymbol{\upsilon})$$

subject to

$$x^+ = Ax + \sum_{i \in \mathbb{I}_{1:M}} B_i \upsilon_i$$
$$\boldsymbol{\upsilon}_i \in \mathbb{U}_i^N$$
$$S_{ji}^{u\prime} x_{ji}(N) = 0 \quad j \in \mathbb{I}_{1:M}$$
$$|\boldsymbol{\upsilon}_i| \le d_i \sum_{j \in \mathbb{I}_{1:M}} \left| x_{ji}(0) \right| \quad \text{if } x_{ji}(0) \in \mathbb{B}_r \; j \in \mathbb{I}_{1:M}$$
$$\boldsymbol{\upsilon}_j = \boldsymbol{\upsilon}_j^p \quad j \in \mathbb{I}_{1:M} \setminus i.$$

The controller iteration is given by

$$\boldsymbol{\upsilon}^{p+1} = \sum_{i \in \mathbb{I}_{1:M}} w_i(\boldsymbol{\upsilon}_1^p, \ldots, \boldsymbol{\upsilon}_i^*, \ldots, \boldsymbol{\upsilon}_M^p)$$

in which $\boldsymbol{\upsilon}_i^* = \boldsymbol{\upsilon}_i^*\left(x(0); \boldsymbol{\upsilon}_j^p, j \in \mathbb{I}_{1:M} \setminus i\right)$. After $\bar{p}$ iterates, we set $\mathbf{u} \leftarrow \boldsymbol{\upsilon}^{\bar{p}}$ and inject $u(0)$ into the plant.

The warm start is generated by purely local information

$$\tilde{\mathbf{u}}_i^+ = \{u_i(1), u_i(2), \ldots, u_i(N-1), 0\} \quad \forall i \in \mathbb{I}_{1:M}.$$

The plantwide cost function then satisfies for any $\bar{p} \ge 0$

$$V(x^+, \mathbf{u}^+) \le V(x, \mathbf{u}) - \sum_{i \in \mathbb{I}_{1:M}} \rho_i \ell_i(x_i, u_i)$$
$$|\mathbf{u}| \le d\,|x| \quad x \in \mathbb{B}_r.$$

Generalizing Assumption 3 to all $i \in \mathbb{I}_{1:M}$, we find that Theorem 9 applies and cooperative MPC of M subsystems is exponentially stable.

Moreover, expressing the M subsystem outputs as

$$y_i = \sum_{j \in \mathbb{I}_{1:M}} C_{ij} x_{ij} \quad i \in \mathbb{I}_{1:M}$$

and generalizing Assumption 10 for $i \in \mathbb{I}_{1:M}$, cooperative MPC for M subsystems satisfies Theorem 13. Finally, for systems with coupled constraints, we can decompose the feasible space such that $\mathbb{U} = \left(\prod_{i \in \mathbb{I}_{1:M}} \mathbb{U}_{Ui}\right) \times \mathbb{U}_C$. Hence, the input augmentation scheme of Section 5 is applicable to plants of M subsystems. Notice that, in general, this approach may lead to augmented inputs for each subsystem that are larger than strictly necessary to achieve optimal control. The most parsimonious augmentation scheme is described elsewhere [22].

7. Example

Consider a plant consisting of two reactors and a separator. A stream of pure reactant A is added to each reactor and converted to the product B by a first-order reaction. The product is lost by a parallel first-order reaction to side product C. The distillate of the separator is split and partially redirected to the first reactor (see Fig. 1).

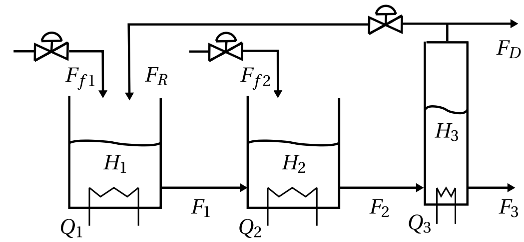

Fig. 1. Two reactors in series with separator and recycle.

The model for the plant is

$$\frac{dH_1}{dt} = \frac{1}{\rho A_1}(F_{f1} + F_R - F_1)$$
$$\frac{dx_{A1}}{dt} = \frac{1}{\rho A_1 H_1}(F_{f1}x_{A0} + F_R x_{AR} - F_1 x_{A1}) - k_{A1}x_{A1}$$
$$\frac{dx_{B1}}{dt} = \frac{1}{\rho A_1 H_1}(F_R x_{BR} - F_1 x_{B1}) + k_{A1}x_{A1} - k_{B1}x_{B1}$$
$$\frac{dT_1}{dt} = \frac{1}{\rho A_1 H_1}(F_{f1}T_0 + F_R T_R - F_1 T_1) - \frac{1}{C_p}(k_{A1}x_{A1}\Delta H_A + k_{B1}x_{B1}\Delta H_B) + \frac{Q_1}{\rho A_1 C_p H_1}$$
$$\frac{dH_2}{dt} = \frac{1}{\rho A_2}(F_{f2} + F_1 - F_2)$$
$$\frac{dx_{A2}}{dt} = \frac{1}{\rho A_2 H_2}(F_{f2}x_{A0} + F_1 x_{A1} - F_2 x_{A2}) - k_{A2}x_{A2}$$
$$\frac{dx_{B2}}{dt} = \frac{1}{\rho A_2 H_2}(F_1 x_{B1} - F_2 x_{B2}) + k_{A2}x_{A2} - k_{B2}x_{B2}$$
$$\frac{dT_2}{dt} = \frac{1}{\rho A_2 H_2}(F_{f2}T_0 + F_1 T_1 - F_2 T_2) - \frac{1}{C_p}(k_{A2}x_{A2}\Delta H_A + k_{B2}x_{B2}\Delta H_B) + \frac{Q_2}{\rho A_2 C_p H_2}$$
$$\frac{dH_3}{dt} = \frac{1}{\rho A_3}(F_2 - F_D - F_R - F_3)$$
$$\frac{dx_{A3}}{dt} = \frac{1}{\rho A_3 H_3}(F_2 x_{A2} - (F_D + F_R)x_{AR} - F_3 x_{A3})$$
$$\frac{dx_{B3}}{dt} = \frac{1}{\rho A_3 H_3}(F_2 x_{B2} - (F_D + F_R)x_{BR} - F_3 x_{B3})$$
$$\frac{dT_3}{dt} = \frac{1}{\rho A_3 H_3}(F_2 T_2 - (F_D + F_R)T_R - F_3 T_3) + \frac{Q_3}{\rho A_3 C_p H_3}$$

in which for all $i \in \mathbb{I}_{1:3}$

$$F_i = k_{vi} H_i \qquad k_{Ai} = k_A \exp\left(-\frac{E_A}{RT_i}\right) \qquad k_{Bi} = k_B \exp\left(-\frac{E_B}{RT_i}\right).$$

The recycle flow and weight percents satisfy

$$F_D = 0.01 F_R \qquad x_{AR} = \frac{\alpha_A x_{A3}}{\bar{x}_3} \qquad x_{BR} = \frac{\alpha_B x_{B3}}{\bar{x}_3}$$
$$\bar{x}_3 = \alpha_A x_{A3} + \alpha_B x_{B3} + \alpha_C x_{C3} \quad x_{C3} = (1 - x_{A3} - x_{B3}).$$

The output and input are denoted, respectively

$$y = \begin{bmatrix} H_1 & x_{A1} & x_{B1} & T_1 & H_2 & x_{A2} & x_{B2} & T_2 & H_3 & x_{A3} & x_{B3} & T_3 \end{bmatrix}$$
$$u = \begin{bmatrix} F_{f1} & Q_1 & F_{f2} & Q_2 & F_R & Q_3 \end{bmatrix}.$$

We linearize the plant model around the steady state defined by Table 1 and derive the following linear discrete-time model with sampling time $\Delta = 0.1$ s

$$x^+ = Ax + Bu \qquad y = x.$$

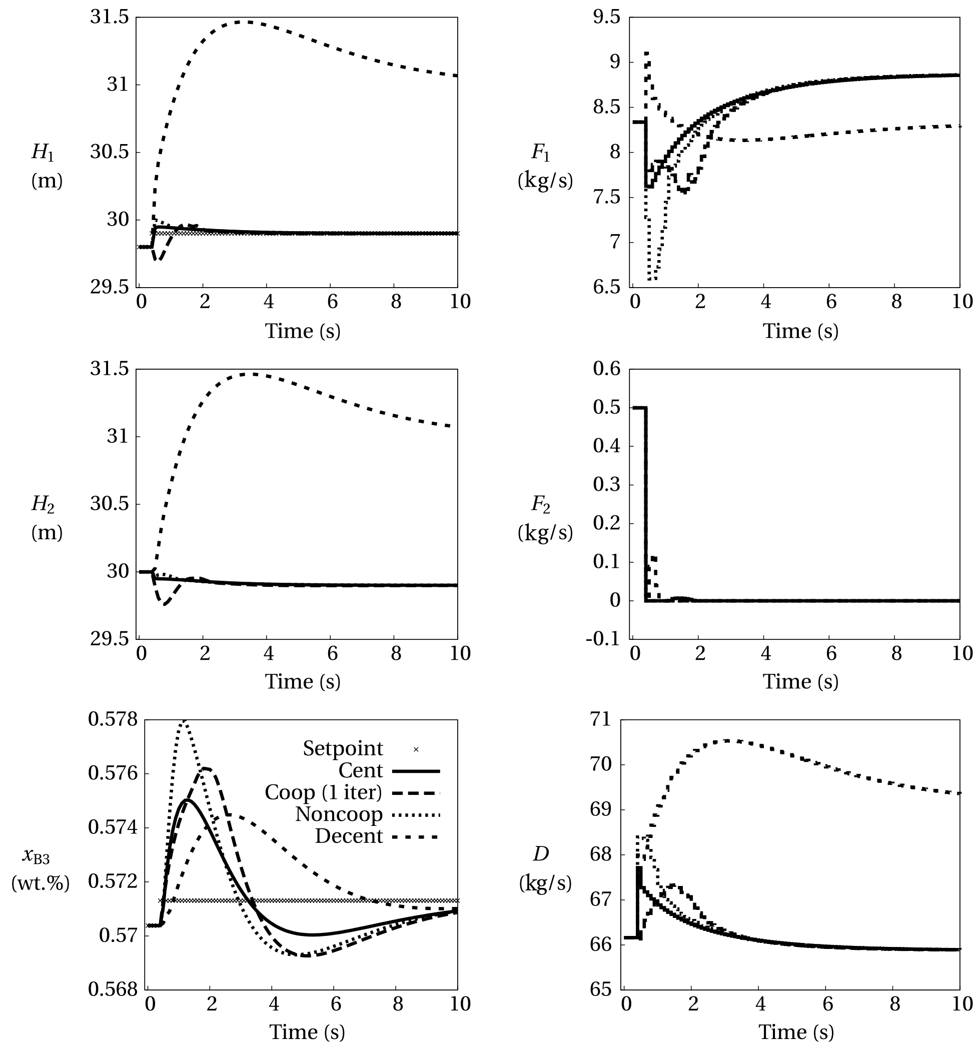

Fig. 2. Performance of reactor and separator example.

Table 1
Steady states and parameters.

Parameter	Value	Units	Parameter	Value	Units
H_1	29.8	m	A_1	3	m^2
x_{A1}	0.542	wt(%)	A_2	3	m^2
x_{B1}	0.393	wt(%)	A_3	1	m^2
T_1	315	K	ρ	0.15	kg/m^3
H_2	30	m	C_p	25	kJ/kg K
x_{A2}	0.503	wt(%)	k_{v1}	2.5	kg/m s
x_{B2}	0.421	wt(%)	k_{v2}	2.5	kg/m s
T_2	315	K	k_{v3}	2.5	kg/m s
H_3	3.27	m	x_{A0}	1	wt(%)
x_{A3}	0.238	wt(%)	T_0	313	K
x_{B3}	0.570	wt(%)	k_A	0.02	1/s
T_3	315	K	k_B	0.018	1/s
F_{f1}	8.33	kg/s	E_A/R	−100	K
Q_1	10	kJ/s	E_B/R	−150	K
F_{f2}	0.5	kg/s	ΔH_A	−40	kJ/kg
Q_2	10	kJ/s	ΔH_B	−50	kJ/kg
F_R	66.2	kg/s	α_A	3.5	
Q_3	10	kJ/s	α_B	1.1	
			α_C	0.5	

7.1. Distributed control

In order to control the separator and each reactor independently, we partition the plant into 3 subsystems by defining

$$y_1 = \begin{bmatrix} H_1 & x_{A1} & x_{B1} & T_1 \end{bmatrix} \qquad u_1 = \begin{bmatrix} F_{f1} & Q_1 \end{bmatrix}$$
$$y_2 = \begin{bmatrix} H_2 & x_{A2} & x_{B2} & T_2 \end{bmatrix} \qquad u_2 = \begin{bmatrix} F_{f2} & Q_2 \end{bmatrix}$$
$$y_3 = \begin{bmatrix} H_3 & x_{A3} & x_{B3} & T_3 \end{bmatrix} \qquad u_3 = \begin{bmatrix} F_R & Q_3 \end{bmatrix}.$$

Following the distributed model derivation in the Appendix (see Appendix B), we form the distributed model for the plant.

7.2. Simulation

Consider the performance of distributed control with the partitioning defined above. The tuning parameters are

$$Q_{yi} = \text{diag}(1, 0, 0, 0.1) \quad \forall i = \mathbb{I}_{1:2} \qquad Q_{y3} = \text{diag}(1, 0, 10^3, 0)$$
$$Q_i = C_i' Q_{yi} C_i + 0.001I \qquad R_i = 0.01I \quad \forall i \in \mathbb{I}_{1:3}.$$

The input constraints are defined in Table 2. We simulate a setpoint change in the output product weight percent x_{B3} at $t = 0.5$ s.

In Fig. 2, the performance of the distributed control strategies are compared to the centralized control benchmark. For this example, noncooperative control is an improvement over decentralized control (see Table 3). Cooperative control with only a single iteration is significantly better than noncooperative control, however, and approaches centralized control as more iteration is allowed.

8. Conclusion

In this paper we present a novel cooperative distributed controller in which the subsystem controllers optimize the same objective function in parallel without the use of a coordinator. The

Table 2
Input constraints.

Parameter	Lower bound	Steady state	Upper bound	Units
F_{f1}	0	8.33	10	kg/s
Q_1	0	10	50	kJ/s
F_{f2}	0	0.5	10	kg/s
Q_2	0	10	50	kJ/s
F_R	0	66.2	75	kg/s
Q_3	0	10	50	kJ/s

Table 3
Performance comparison.

	Cost	Performance loss (%)
Centralized MPC	1.76	0
Cooperative MPC (10 iterates)	1.94	9.88
Cooperative MPC (1 iterate)	3.12	76.8
Noncooperative MPC	4.69	166
Decentralized MPC	185	1.04 e+ 04

control algorithm is equivalent to a suboptimal centralized controller, allowing the distributed optimization to be terminated at any iterate before convergence. At convergence, the feedback is Pareto optimal. We establish exponential stability for the nominal case and for perturbation by a stable state estimator. For plants with sparsely coupled constraints, the controller can be extended by repartitioning the decision variables to maintain Pareto optimality.

We make no restrictions on the strength of the dynamic coupling in the network of subsystems, offering flexibility in plantwide control design. Moreover, the cooperative controller can improve performance of plants over traditional decentralized control and noncooperative control, especially for plants with strong open-loop interactions between subsystems. A simple example is given showing this performance improvement.

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Appendix A. Further proofs

Proof of Lemma 5. By assumption, the initial guess is feasible. Because $\mathbb{U}_1$ and $\mathbb{U}_2$ are convex, the convex combination (3.7) with $p = 0$ implies $(\upsilon_1^1, \upsilon_2^1)$ is feasible. Feasibility for $p > 1$ follows by induction. □

Proof of Lemma 6. For every $p \geq 0$, the cost function satisfies the following

$$V(x(0), \upsilon^{p+1}) = V(x(0), w_1(\upsilon_1^*, \upsilon_2^p) + w_2(\upsilon_1^p, \upsilon_2^*))$$

$$\leq w_1 V(x(0), (\upsilon_1^*, \upsilon_2^p)) + w_2 V(x(0), (\upsilon_1^p, \upsilon_2^*)) \quad \text{(A.1a)}$$

$$\leq w_1 V(x(0), (\upsilon_1^p, \upsilon_2^p)) + w_2 V(x(0), (\upsilon_1^p, \upsilon_2^p)) \quad \text{(A.1b)}$$

$$\leq V(x(0), \upsilon^p).$$

The first equality follows from (3.7). The inequality (A.1a) follows from convexity of $V(\cdot)$. The next inequality (A.1b) follows from the optimality of υ_i^* $\forall i \in \mathbb{I}_{1:2}$, and the final line follows from $w_1 + w_2 = 1$. Because the cost is bounded below, it converges. □

Proof of Lemma 7. We give a proof that requires only closedness (not compactness) of $\mathbb{U}_i$, $i \in \mathbb{I}_{1:2}$. From Lemma 6, the cost converges, say to $\underline{V}$. Since V is quadratic and strongly convex, its sublevel sets $\text{lev}_{\leq a}(V)$ are compact and bounded for all a. Hence, all iterates belong to the compact set $\text{lev}_{\leq V(\upsilon^0)}(V) \cap \mathbb{U}$, so there is at least one accumulation point. Let $\bar{\upsilon}$ be any such accumulation point, and choose a subsequence $\mathscr{P} \subset \{1, 2, 3, \ldots\}$ such that $\{\upsilon^p\}_{p\in\mathscr{P}}$ converges to $\bar{\upsilon}$. We obviously have that $V(x(0), \bar{\upsilon}) = \underline{V}$, and moreover that

$$\lim_{p\in\mathscr{P}, p\to\infty} V(x(0), \upsilon^p) = \lim_{p\in\mathscr{P}, p\to\infty} V(x(0), \upsilon^{p+1}) = \underline{V}. \quad \text{(A.2)}$$

By strict convexity of V and compactness of $\mathbb{U}_i$, $i \in \mathbb{I}_{1:2}$, the minimizer of $V(x(0), \cdot)$ is attained at a unique point $\mathbf{u}^0 = (\mathbf{u}_1^0, \mathbf{u}_2^0)$. By taking limits in (A.1) as $p \to \infty$ for $p \in \mathscr{P}$, and using $w_1 > 0$, $w_2 > 0$, we deduce directly that

$$\lim_{p\in\mathscr{P}, p\to\infty} V(x(0), (\upsilon_1^*(\upsilon_2^p), \upsilon_2^p)) = \underline{V} \quad \text{(A.3a)}$$

$$\lim_{p\in\mathscr{P}, p\to\infty} V(x(0), (\upsilon_1^p, \upsilon_2^*(\upsilon_1^p))) = \underline{V}. \quad \text{(A.3b)}$$

We suppose for contradiction that $\underline{V} \neq V(x(0), \mathbf{u}^0)$ and thus $\bar{\upsilon} \neq \mathbf{u}^0$. Because $V(x(0), \cdot)$ is convex, we have

$$\nabla V(x(0), \bar{\upsilon})'(\mathbf{u}^0 - \bar{\upsilon}) \leq \Delta V := V(x(0), \mathbf{u}^0) - V(x(0), \bar{\upsilon}) < 0$$

where $\nabla V(x(0), \cdot)$ denotes the gradient of $V(x(0), \cdot)$. It follows immediately that either

$$\nabla V(x(0), \bar{\upsilon})' \begin{bmatrix} \mathbf{u}_1^0 - \bar{\upsilon}_1 \\ 0 \end{bmatrix} \leq (1/2)\Delta V \quad \text{or} \quad \text{(A.4a)}$$

$$\nabla V(x(0), \bar{\upsilon})' \begin{bmatrix} 0 \\ \mathbf{u}_2^0 - \bar{\upsilon}_2 \end{bmatrix} \leq (1/2)\Delta V. \quad \text{(A.4b)}$$

Suppose first that (A.4a) holds. Applying Taylor's theorem to V

$$\begin{aligned} &V(x(0), (\upsilon_1^p + \epsilon(\mathbf{u}_1^0 - \upsilon_1^p), \upsilon_2^p)) \\ &\quad = V(x(0), \upsilon^p) + \epsilon \nabla V(x(0), \upsilon^p)' \begin{bmatrix} \mathbf{u}_1^0 - \upsilon_1^p \\ 0 \end{bmatrix} \\ &\qquad + \frac{1}{2}\epsilon^2 \begin{bmatrix} \mathbf{u}_1^0 - \upsilon_1^p \\ 0 \end{bmatrix}' \nabla^2 V(x(0), \upsilon_1^p \\ &\qquad + \gamma\epsilon(\mathbf{u}_1^0 - \upsilon_1^p), \upsilon_2^p) \begin{bmatrix} \mathbf{u}_1^0 - \upsilon_1^p \\ 0 \end{bmatrix} \\ &\quad \leq \underline{V} + (1/4)\epsilon\Delta V + \beta\epsilon^2 \end{aligned} \quad \text{(A.5)}$$

in which $\gamma \in (0, \epsilon)$. (A.5) applies for all $p \in \mathscr{P}$ sufficiently large, for some β independent of ϵ and p. By fixing ϵ to a suitably small value (certainly less than 1), we have both that the right-hand side of (A.5) is strictly less than $\underline{V}$ and that $\upsilon_1^p + \epsilon(\mathbf{u}_1^0 - \upsilon_1^p) \in \mathbb{U}_1$. By taking limits in (A.5) and using (A.3) and the fact that $\upsilon_1^*(\upsilon_2^p)$ is optimal for $V(x(0), (\cdot, \upsilon_2^p))$ in $\mathbb{U}_1$, we have

$$\begin{aligned} \underline{V} &= \lim_{p\in\mathscr{P}, p\to\infty} V(x(0), (\upsilon_1^*(\upsilon_2^p), \upsilon_2^p)) \\ &\leq \lim_{p\in\mathscr{P}, p\to\infty} V(x(0), (\upsilon_1^p + \epsilon(\mathbf{u}_1^0 - \upsilon_1^p), \upsilon_2^p)) \\ &< \underline{V} \end{aligned}$$

giving a contradiction. By identical logic, we obtain the same contradiction from (A.4b). We conclude that $\underline{V} = V(x(0), \mathbf{u}^0)$ and thus $\bar{\upsilon} = \mathbf{u}^0$. Since $\bar{\upsilon}$ was an arbitrary accumulation point of the sequence $\{\upsilon^p\}$, and since this sequence is confined to a compact set, we conclude that the whole sequence converges to $\mathbf{u}^0$. □

Proof of Corollary 14. We first note that: $|x(k)| \leq |\hat{x}(k)| + |e(k)| \leq \sqrt{2}|(\hat{x}(k), e(k))|$. From Theorem 13 we can write:

$$|x(k)| \leq \sqrt{2}\alpha \left|(\hat{x}(0), e(0))\right| \gamma^k \leq \bar{\alpha} \left|\hat{x}(0) + e(0)\right| \gamma^k$$

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with $\bar{\alpha} = \sqrt{2}\alpha$, which concludes the proof by noticing that $x(0) = \hat{x}(0) + e(0)$. □

Proof of Lemma 15. Because $\hat{V}(\cdot)$ is convex and bounded below, the proof follows from Lemma 7 and from noticing that the point $\mathbf{u}^0 = (\hat{E}_1\hat{\mathbf{u}}_1^0, \hat{E}_2\hat{\mathbf{u}}_2^0)$, with $\hat{\mathbf{u}}_i^0 = \lim_{p\to\infty} \hat{\mathbf{v}}_i$, $i \in \mathbb{I}_{1:2}$, is Pareto optimal. □

Appendix B. Deriving the distributed model

Consider the possibly nonminimal centralized model

$$x^+ = Ax + \sum_{j\in\mathbb{I}_{1:M}} B_j u_j \tag{B.1}$$
$$y_i = C_i x \quad \forall i \in \mathbb{I}_{1:M}.$$

For each input/output pair (u_j, y_i) we transform the triple (A, B_j, C_i) into its Kalman canonical form [23, p.270]

$$\begin{bmatrix} z_{ij}^{oc} \\ z_{ij}^{\bar{o}c} \\ z_{ij}^{o\bar{c}} \\ z_{ij}^{\bar{o}\bar{c}} \end{bmatrix}^+ = \begin{bmatrix} A_{ij}^{oc} & 0 & A_{ij}^{oc\bar{c}} & 0 \\ A_{ij}^{\bar{o}oc} & A_{ij}^{\bar{o}c} & A_{ij}^{\bar{o}co\bar{c}} & A_{ij}^{\bar{o}c\bar{c}} \\ 0 & 0 & A_{ij}^{o\bar{c}} & 0 \\ 0 & 0 & A_{ij}^{\bar{o}\bar{c}o} & A_{ij}^{\bar{o}\bar{c}} \end{bmatrix} \begin{bmatrix} z_{ij}^{oc} \\ z_{ij}^{\bar{o}c} \\ z_{ij}^{o\bar{c}} \\ z_{ij}^{\bar{o}\bar{c}} \end{bmatrix} + \begin{bmatrix} B_{ij}^{oc} \\ B_{ij}^{\bar{o}c} \\ 0 \\ 0 \end{bmatrix} u_j$$

$$y_{ij} = \begin{bmatrix} C_{ij}^{oc} & 0 & C_{ij}^{o\bar{c}} & 0 \end{bmatrix} \begin{bmatrix} z_{ij}^{oc} \\ z_{ij}^{\bar{o}c} \\ z_{ij}^{o\bar{c}} \\ z_{ij}^{\bar{o}\bar{c}} \end{bmatrix} \qquad y_i = \sum_{j\in\mathbb{I}_{1:M}} y_{ij}.$$

The vector z_{ij}^{oc} captures the modes of A that are both observable by y_i and controllable by u_j. The distributed model is then

$$A_{ij} \leftarrow A_{ij}^{oc} \qquad B_{ij} \leftarrow B_{ij}^{oc} \qquad C_{ij} \leftarrow C_{ij}^{oc} \qquad x_{ij} \leftarrow z_{ij}^{oc}.$$

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