Introduction to Game Semantics and Logical Games

Charlie Murphy

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Overview

- Finite Games
- Game Semantics
- Strategy Synthesis
- Infinite Games

Finite Games

Two-person Perfect Information Finite Games

Finite Games

- Between two players I and II
- Zero-sum: Player I wins iff Player II loses and vice versa
- Both players have perfect knowledge:
  - Of how the game is played
  - Of the moves played so far

- Examples: Chess, Checkers, Tic-Tac-Toe, Reversi, Go
- Negative Examples: Backgammon, Yahtzee, Rock-Paper-Scissors, Stratego

[Khomskii, 2010]

Finite Games

- Let $n \in \mathbb{N}$ and $A \subseteq \mathbb{N}^{2n}$ then $G(A)_n$ is a game:
  - Two players: I and II
    - Take turns choosing a natural number
    - Both know A the set of wins for player I (losses for player II)
    - x_i and y_i represents the ith move by I and II respectively
  - After n moves by each player the game ends:

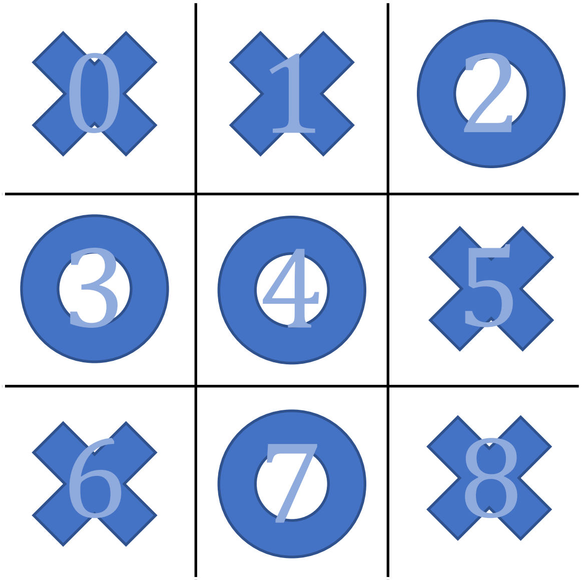
I	x_0		x_1		$\cdots$		x_{n-1}	
II		y_0		y_1		$\cdots$		y_{n-1}

  - $s = x_0 y_0 x_1 y_1 \ldots x_{n-1} y_{n-1}$ is a play of $G(A)_n$
  - Player I wins iff s $\in A$ and Player II wins iff $s \notin A$

[Khomskii, 2010]

Tic-Tac-Toe

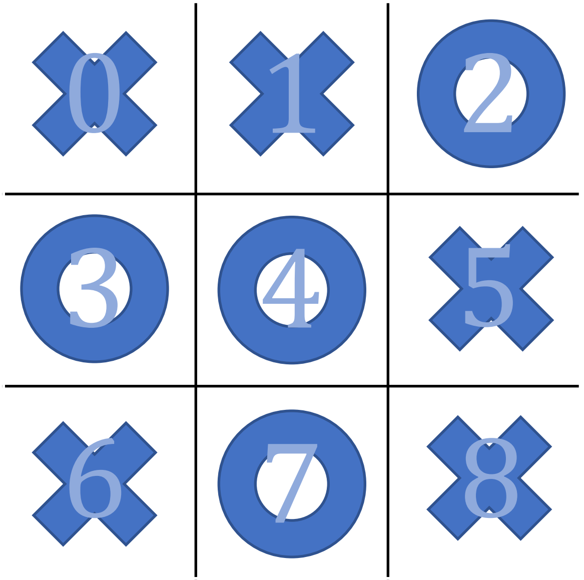
- Two players X and O.
- Encode moves as $0-9$.
- $A = Legal \backslash Wins_O \cup Illegal_O$
- $A' = Legal \times Wins_X \cup Illegal_O$
- X wins iff X wins $G(A)$ and $G(A')$
- O wins iff O wins $G(A)$ and $G(A')$
- Otherwise X and O draw



X	0		6		5		1		8	
O		4		3		7		2		9

Strategies

- Given a game, $G(A)$,
  - What move should I or II make next?
  - A function from partial plays to moves.
- Tic-Tac-Toe
  - Strategy, σ, of player X
  - Strategy, τ, of player O



$\sigma(\epsilon) = 0 \quad \sigma(04) = 6 \quad \sigma(0463) = 5 \quad \sigma(046357) = 1 \quad \sigma(04635712) = 8$

$\tau(0) = 4 \quad \tau(046) = 3 \quad \tau(04635) = 7 \quad \tau(0463571) = 2 \quad \tau(046357128) = 9$

Strategies

- Given a game $G(A)_n$:
  - A Strategy for player I is a function, σ:
    - $\sigma : \{s \in \bigcup_{m<2n} \mathbb{N}^m : |s| \text{ is even}\} \to \mathbb{N}$
  - A strategy for player II is a function, τ:
    - $\tau : \{s \in \bigcup_{m<2n} \mathbb{N}^m : |s| \text{ is odd}\} \to \mathbb{N}$
  - $\sigma * t$ is a play where I plays with σ and II plays t
  - $s * \tau$ is a play where I plays with s and II plays τ
  - $plays(\sigma)_n = \{\sigma * t : t \in \mathbb{N}^n\}$
  - $plays(\tau)_n = \{s * \tau : s \in \mathbb{N}^n\}$

[Khomskii, 2010]

Winning Strategies

- Given a game $G(A)_n$:
  - σ is a winning strategy iff $plays(\sigma)_n \subseteq A$
  - τ is a winning strategy iff $plays(\tau)_n \subseteq \mathbb{N}^{2n} \backslash A$

Theorem: For all games, $G(A)$, player I and II cannot both have a winning strategy.

Proof.

Suppose not.

I has a winning strategy σ and II a winning strategy τ.

Let $s = \sigma * \tau$ be the play where I follows σ and II follows τ.

Necessarily $s \in A$ and $s \notin A$, a contradiction. **QED**

Determinacy

- A game G(A) is determined iff Player I or II has a winning strategy
  - There is at most one winner
  - Is there at least one winner?

Theorem: For all finite games, $G(A)_n$, player I or II must have a winning strategy.

Proof. Player I has a winning strategy iff

$$\exists x_0 \forall y_0 \ldots \exists x_{n-1} \forall y_{n-1}.\, x_0 y_0 \ldots x_{n-1}\, y_{n-1} \in A$$

Suppose I does not have a winning strategy:

$$\neg(\exists x_0 \forall y_0 \ldots \exists x_{n-1} \forall y_{n-1}.\, (x_0 y_0 \ldots x_{n-1}\, y_{n-1} \in A))$$

$$\ldots$$

$$\exists x_0 \forall y_0 \ldots \exists x_{n-1} \forall y_{n-1}.\, (x_0 y_0 \ldots x_{n-1}\, y_{n-1} \notin A)$$

Player II must have a winning strategy. **QED**

Game Semantics

Game Semantics

- Given a sentence $S \in L$ and a model, M, of non-logical symbols:
  - Define a game, $G(S;\ M)$ to show if $S \in L$
- First Order Logic:
  - $G(S;\ M)$ is a game between Myself (init. verifier) and Nature (init. falsifier)
  - $G(S;\ M)$ is played using the following rules:

R $\vee$. $G(S_1 \vee S_2; M)$ verifier chooses to continue as $G(S_1; M)$ or $G(S_2; M)$

R $\wedge$. $G(S_1 \wedge S_2; M)$ falsifier chooses to continue as $G(S_1; M)$ or $G(S_2; M)$

R $\exists$. $G(\exists x. S; M)$ verifier chooses $c \in do(M)$ and continues as $G(S[c/\mathrm{x}]; M)$

R $\forall$. $G(\forall x. S; M)$ falsifier chooses $c \in do(M)$ and continues as $G(S[c/\mathrm{x}]; M)$

R $\neg$. $G(\neg S; M)$ falsifier and verifier swap roles and play $G(S; M)$

R atom. $G(c_a; M)$ the current verifier wins if c_a interpreted in M is true otherwise falsifier wins

[Hintikka, 1982]

Game Semantics

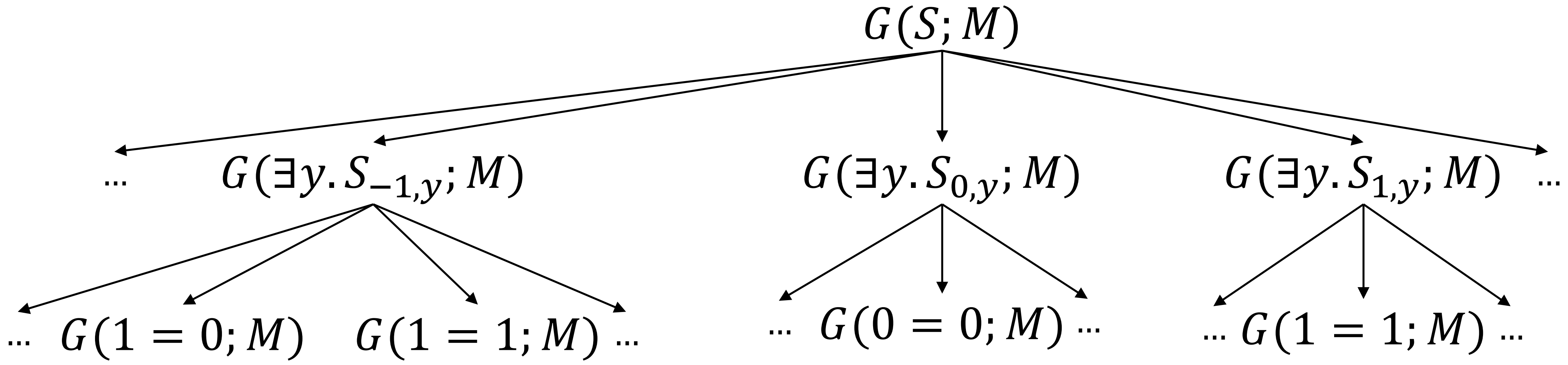
- Given a sentence $S \in L$ and a model, M, of non-logical symbols:
  - Define a game, $G(S;\ M)$ to show if M $\vDash S$
- First Order Logic:
  - $G(S;\ M)$ is a game between Myself (init. verifier) and Nature (init. falsifier)
  - $G(S;\ M)$ is played using the following rules:
    - $R \lor.$, $R \land.$, $R \exists.$, $R \forall.$, $R \neg.$, and R atom.
  - $M \vDash S$ iff Myself wins $G(S; M)$

[Hintikka, 1982]

First Order Logic

$$M = \mathbb{Z} \cup \{abs, <, +, =\}$$

$$S = \forall x \exists y.\, \underbrace{abs(x) = y}_{S_{x,y}}$$



First Order Logic

$$M = \mathbb{Z} \cup \{abs, <, +, =\}$$

$$S = \forall x \exists y.\, \underbrace{abs(x) = y}_{S_{x,y}}$$

$$G(S; M)$$

↓ Falsifier Chooses c

$$G(\exists y.\, S_{c,y}; M)$$

↓ Verifier Chooses $abs(c)$

$$G(abs(c) = abs(c); M)$$

Strategy Synthesis

Strategy Synthesis

- Given a game $G(A)$:
  - Can we know if a player has some winning strategy?
  - Can we produce this strategy?
- Logical Games:
  - Defined using logical formulae
  - Strategy synthesis corresponds to
    - Functional & Reactive Synthesis
    - Adversarial Planning
    - Modular Verification
    - Branching-Time Verification

[Farzan & Kincaid, 2016]

Linear Arithmetic Satisfiability Games

Satisfiability Games

- Given a sentence, φ, in Linear Arithmetic, construct a game, $G(\varphi)$:
  - Two players SAT and UNSAT take turns instantiating quantifiers
  - SAT controls existentials and wants to prove φ
  - UNSAT controls universals and wants to disprove φ
  - SAT wins a play $c_0 \ldots c_n$ if $[x_0 \mapsto c_0, \ldots, x_n \mapsto c_n] \models F$
  - If SAT has a winning strategy φ is satisfiable

φ, in some theory (e.g. Linear Arithmetic)

$$s, t \in Term ::= c \mid x \mid s + t \mid c \cdot t$$

$$F, G \in Formula ::= t < 0 \mid t = 0 \mid F \wedge G \mid F \vee G$$

$$\varphi_n ::= Q_0 x_0 \ldots Q_n x_n . F \qquad Q_i \in \{\forall, \exists\}$$

φ is a sentence if it has no free variables

[Farzan & Kincaid, 2016]

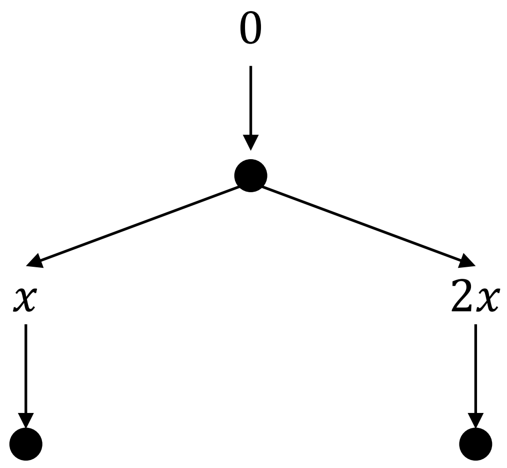
Strategy Improvement

- We first compute a strategy skeleton for SAT

[Farzan & Kincaid, 2016]

Skeleton Strategy

$$\exists w \forall x \exists y \forall z. (y < 1 \lor 2w < y) \land (z < y \lor x < z)$$



[Farzan & Kincaid, 2016]

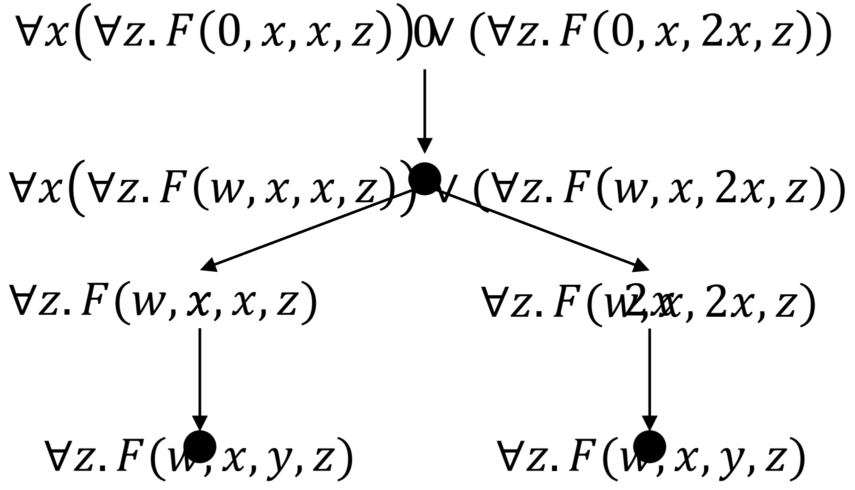
Strategy Improvement

- We first compute a strategy skeleton for SAT
- Is this strategy winning?

[Farzan & Kincaid, 2016]

Winning Condition

$$\exists w \forall x \exists y \forall z. \underbrace{(y < 1 \lor 2w < y) \land (z < y \lor x < z)}_{F(w,x,y,z)}$$



[Farzan & Kincaid, 2016]

Strategy Improvement

- We first compute a strategy skeleton for SAT
- Is this strategy winning?
  - Yes, return winning strategy skeleton
  - No, compute UNSAT's counter-strategy

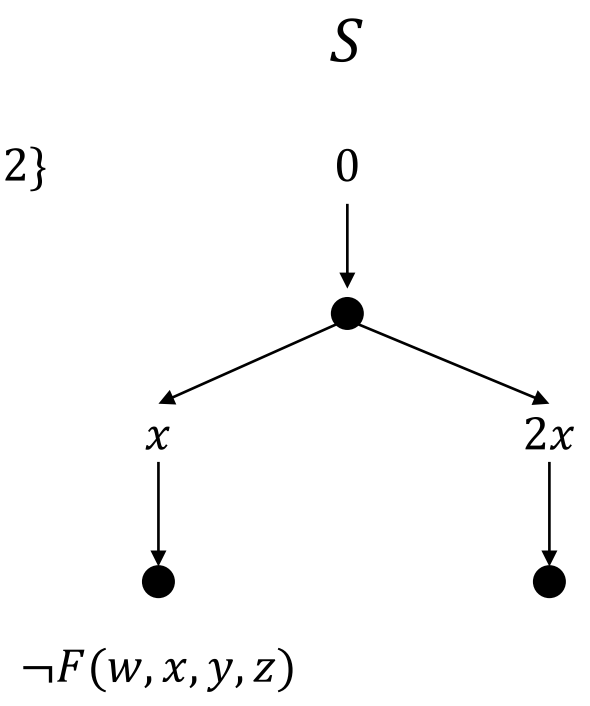
[Farzan & Kincaid, 2016]

Counter Strategy

$$\neg win \text{ is sat iff } \exists \underline{x}. \left(\neg F(0, \underline{x}, \underline{x}, \underline{z_1}) \vee \neg F(0, 2\underline{x}, 2\underline{x}, \underline{z_2}) \right)$$

$$M = \{\underline{x} \mapsto -2, \underline{z_1} \mapsto -2, \underline{z_2} \mapsto -3\}$$

$\boldsymbol{M^\pi = \{w \mapsto 0, x \mapsto -2, y \mapsto -2, z \mapsto -2\}}$



[Farzan & Kincaid, 2016]

Model-based Term Selection

$$select(M, X, F) = \begin{cases} eq(M, x, F) & \text{if } EQ(M, x, F) \neq \emptyset \\ \frac{1}{2}(lub(M, x, F) + glb(M, x, F)) & \text{if } UB(M, x, F) \neq \emptyset \\ & \text{and } LB(\text{M, x, F}) \neq \emptyset \\ lub(M, x, F) - 1 & \text{if } UB(M, x, F) \neq \emptyset \\ glb(M, x, F) + 1 & \text{if } LB(M, \text{x, F}) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$EQ(M, x, F) = \{s : x = s \in F_x \wedge [\![x]\!]^M = [\![s]\!]^M\}$$

$$UB(M, x, F) = \{s : x < s \in F_x \wedge [\![x]\!]^M < [\![s]\!]^M\}$$

$$UB(M, x, F) = \{s : s > x \in F_x \wedge [\![s]\!]^M < [\![x]\!]^M\}$$

Properties of ***select***

$$M \vDash F \Rightarrow M \vDash F[select(M, x, F)/x]$$

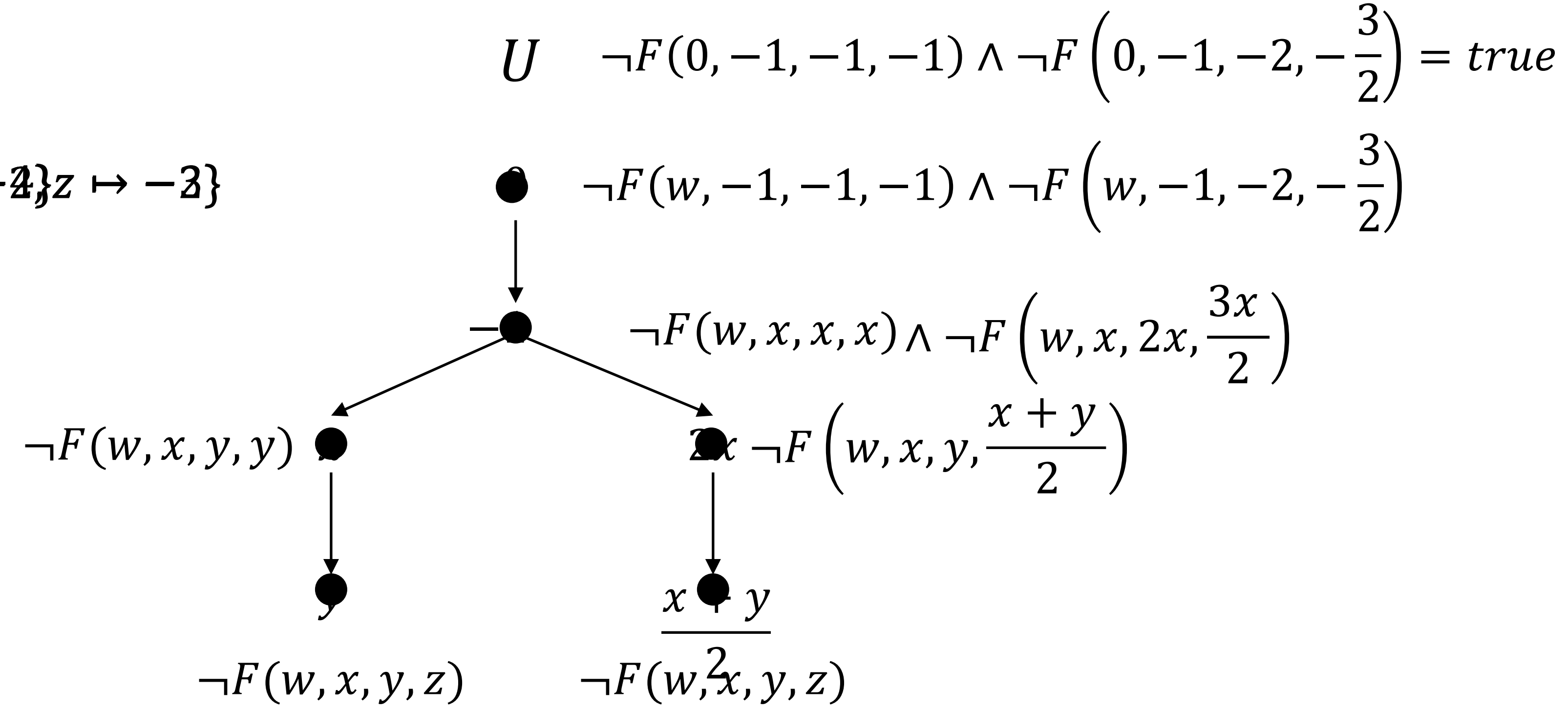
$\{select(M, x, F) : M \vDash F\}$ is finite

Counter Strategy

$\neg win$ is sat iff $\exists M. M \vDash \neg F\left(0, \underline{x}, \underline{x}, \underline{z_1}\right) \wedge \neg F\left(0, \underline{x}, 2\underline{x}, \underline{z_2}\right)$

$M = \{\underline{x} \mapsto -2, \underline{z_1} \mapsto -2, \underline{z_2} \mapsto -3\}$

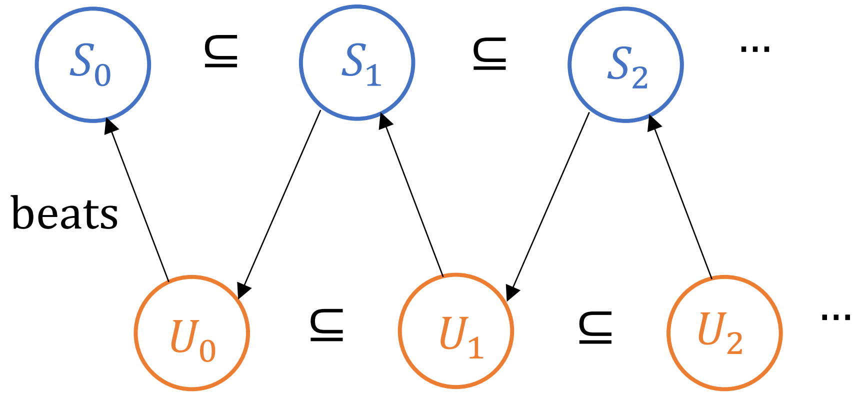
$\boldsymbol{M^{\pi} = \{w \mapsto 0, x \mapsto -2, y \mapsto -4, z \mapsto -3\}}$



[Farzan & Kincaid, 2016]

Strategy Improvement

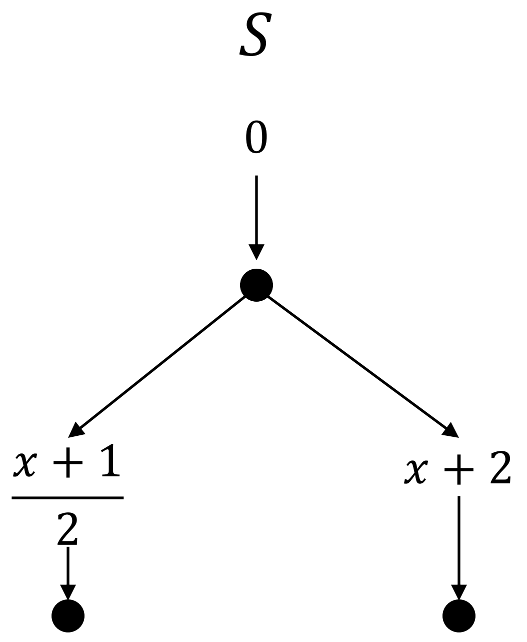
- We first compute a strategy skeleton for SAT
- Is this strategy winning?
  - Yes, return winning strategy skeleton
  - No, compute UNSAT's counter-strategy, trade roles, and repeat



[Farzan & Kincaid, 2016]

Winning Strategy Skeleton

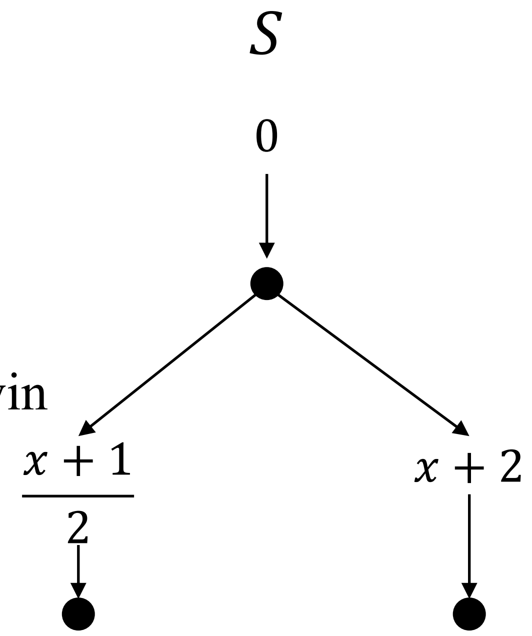
$$\exists w \forall x \exists y \forall z. (y < 1 \lor 2w < y) \land (z < y \lor x < z)$$



[Farzan & Kincaid, 2016]

Strategy Synthesis

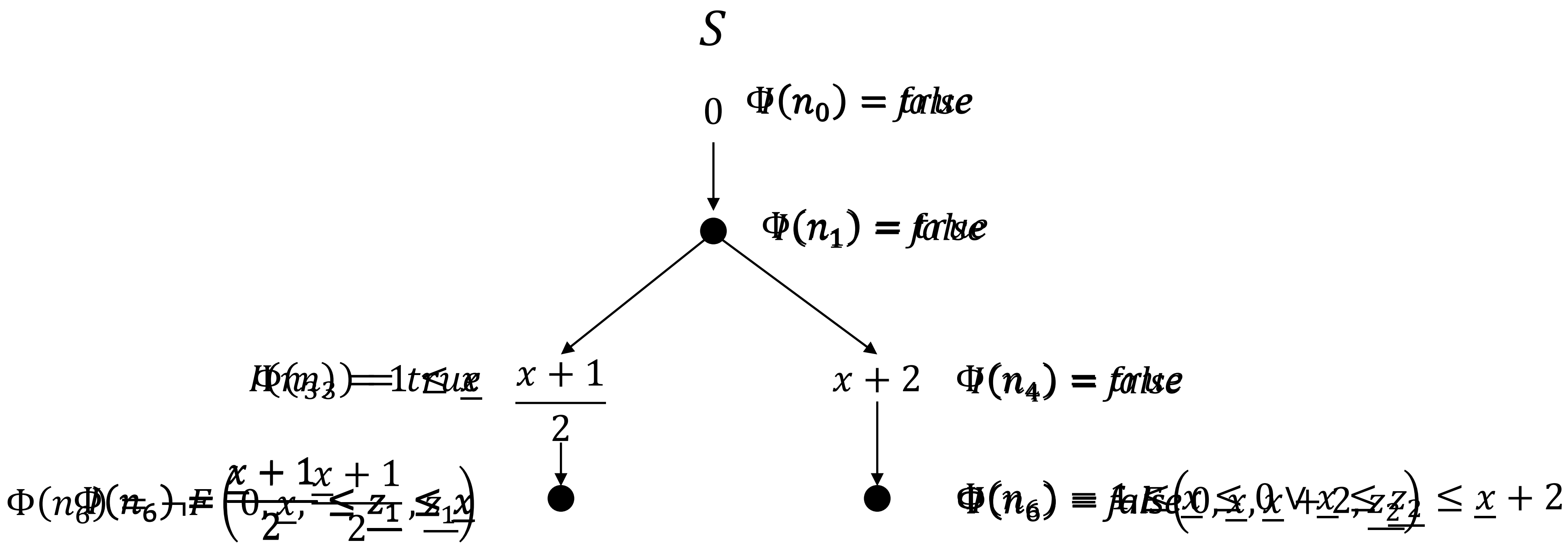
- A winning strategy skeleton isn't a strategy:
  - Some strategy conforming skeleton, S, is winning
  - Can we compute a winning strategy from S? **YES**
- Label skeleton node's with deterministic guards
  - Computed using Tree Interpolants
    - Interpolants represent plays reaching a node where UNSAT may win



[Farzan & Kincaid, 2018]

Tree Interpolation

$$\exists w \forall x \exists y \forall z.\underbrace{(y < 1 \vee 2w < y) \wedge (z < y \vee x < z)}_{F(w,x,y,z)}$$



Strategy Synthesis

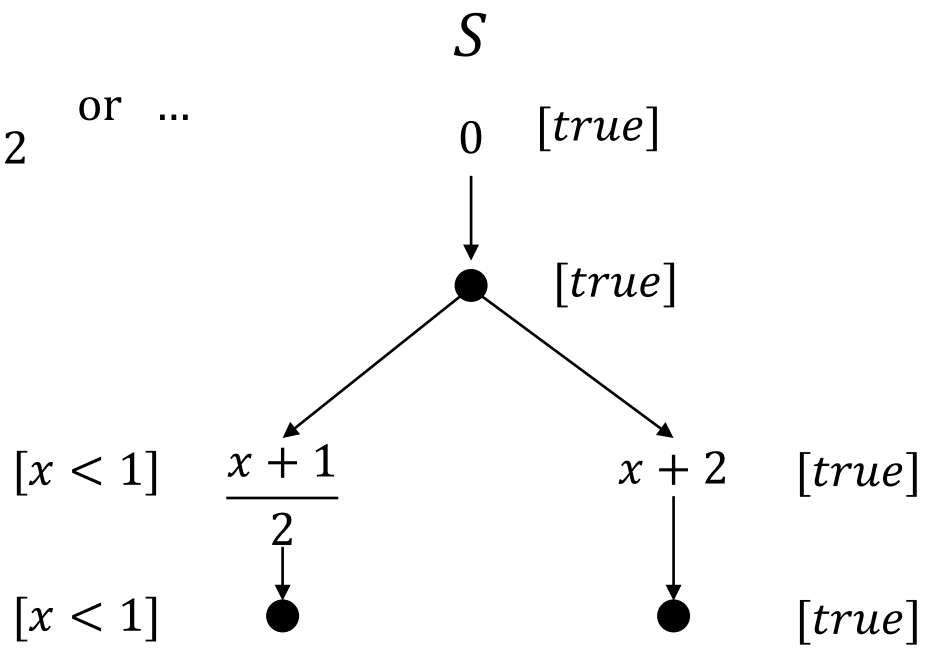
$$\exists w \forall x \exists y \forall z. (y < 1 \vee 2w < y) \wedge (z < y \vee x < z)$$

$\sigma(\epsilon) = 0$
$\sigma(wx) = x + 2$

or

$\sigma'(\epsilon) = 0$
$\sigma'(wx) = \text{if } x < 1 \text{ then } \frac{x+1}{2} \text{ else } x + 2$

or ...



Infinite Games

Infinite Games

- Given $A \subseteq \mathbb{N}^{\mathbb{N}}$, $G(A)$ is an infinite game:
  - Two players: I and II
    - Take turns choosing a natural number (forever)
    - Both know A the set of wins for player I (losses for player II)
    - x_i and y_i represents the ith move by I and II respectively

I	x_0		x_1		$\cdots$		x_i		$\cdots$	
II		y_0		y_1		$\cdots$		y_i		$\cdots$

  - $s = x_0 y_0 x_1 y_1 \ldots$ is a play of $G(A)$
  - Player I wins iff $s \in A$ and Player II wins iff $s \notin A$

[Khomskii, 2010]

Strategies

- Given an infinite game $G(A)$:
  - A Strategy for player I is a function, σ:
    - $\sigma : \{s \in \mathbb{N}^*: |s| \text{ is even}\} \to \mathbb{N}$
  - A strategy for player II is a function, τ:
    - $\tau : \{s \in \mathbb{N}^*: |s| \text{ is odd}\} \to \mathbb{N}$
  - $\sigma * t$ is a play where I plays with σ and II plays t
  - $s * \tau$ is a play where I plays with s and II plays τ
  - $plays(\sigma) = \{\sigma * t : t \in \mathbb{N}^{\mathbb{N}}\}$
  - $plays(\tau) = \{s * \tau : s \in \mathbb{N}^{\mathbb{N}}\}$

[Khomskii, 2010]

Winning Strategies

- Given a game $G(A)$:
  - σ is a winning strategy iff $plays(\sigma) \subseteq A$
  - τ is a winning strategy iff $plays(\tau) \subseteq \mathbb{N}^{\mathbb{N}} \setminus A$

Theorem: For all games, $G(A)$, player I and II cannot both have a winning strategy.

Proof.

Suppose not.

I has a winning strategy σ and II a winning strategy τ.

Let $s = \sigma * \tau$ be the play where I follows σ and II follows τ.

Necessarily $s \in A$ and $s \notin A$, a contradiction. **QED**

Determinacy

- A game G(A) is determined iff Player I or II has a winning strategy
  - There is at most one winner
  - Is there at least one winner?

Theorem: There exists an infinite game that is not determined.

  - Finitely decided games:
    - $G(A)$ is finitely decided iff $\forall s \in A. \exists n. \{s' : \forall i < n. s'_i = s_i\} \subseteq A$

Theorem: All finitely decided games are determined.

[Khomskii, 2010]

Strategy Synthesis
(for infinite games)

Reachability Games

- $G(init, reach, safe)$ is an infinite game:
  - Two players REACH and SAFE alternate picking $\mathbb{Q}^d$ positions
  - REACH starts by picking r_0 satisfying $init$
  - SAFE moves from REACH's choice, r_i, to any state s_i satisfying $safe(r_i, s_i)$
  - REACH then continues play choosing r_{i+1} satisfying $reach(s_i, r_{i+1})$
  - The first player to make an illegal move loses.
  - SAFE wins all games where both players only make legal moves.
  - The game is determined
  - Deciding which player wins is undecidable

[Farzan & Kincaid, 2018]

Strategy Synthesis

- Key Idea: use bounded games to produce satisfiability games.

  $\exists x_1 \forall y_1 \ldots \exists x_n \forall y_n. init(x_1) \land safe(x_1, y_1) \Rightarrow unroll(1, n-1)$

  $unroll(k, 0) \stackrel{\text{def}}{=} false$

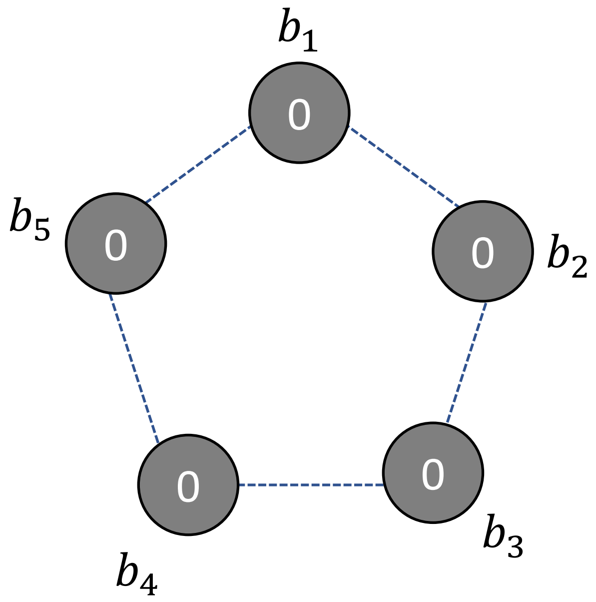
  $unroll(k, d) \stackrel{\text{def}}{=} reach(y_k, x_{k+1}) \land (safe(x_{k+1}, y_{k+1}) \Rightarrow unroll(k+1, d-1))$

  - If REACH (SAT) wins the bounded game
    - REACH wins the unbounded game
    - Any extension of the finite strategy is a winning strategy
  - If SAFE (UNSAT) wins the bounded game
    - Attempt to generalize strategy to infinite games

[Farzan & Kincaid, 2018]

Cinderella-Stepmother Game

- Two players Cinderella and her Stepmother.
- Each round
  - Stepmother adds 1L of water to a buckets (3L capacity)
  - Cinderella can empty two adjacent buckets
- Cinderella wins if no bucket overflows



[Farzan & Kincaid, 2018]

Safety Tree

$n_1: b \leq 3,3,1,1,1 \land (b_3 + b_5) \leq 1$

$true: \langle 0,0, b_3, b_4, b_5 \rangle$

$n_2: b \leq 3,3,2,2,2 \land (b_3 + b_5) \leq 2$

$true: \langle 0,0, b_3, b_4, b_5 \rangle$

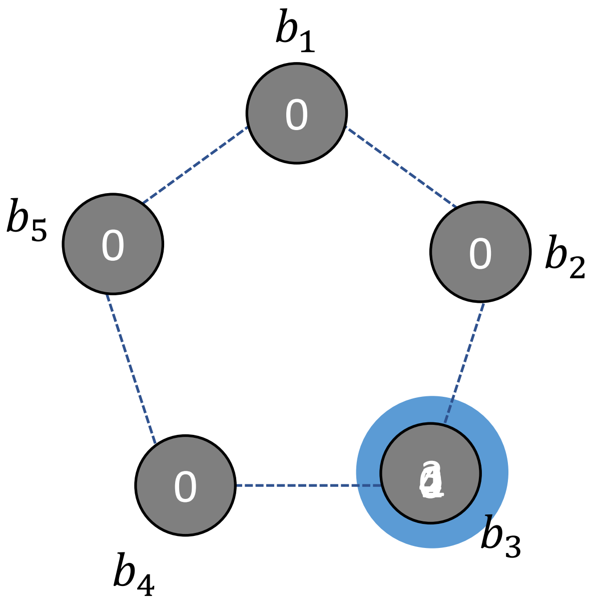
$n_3: b \leq 1,1,3,3,3 \land (b_3 + b_5) \leq 3$

$true: \langle 0,0, b_3, b_4, b_5 \rangle$

$n_3: b \leq 1,1,4,4,4 \land (b_3 + b_5) \leq 4$ ✖

[Farzan & Kincaid, 2018]

Cinderella-Stepmother Game



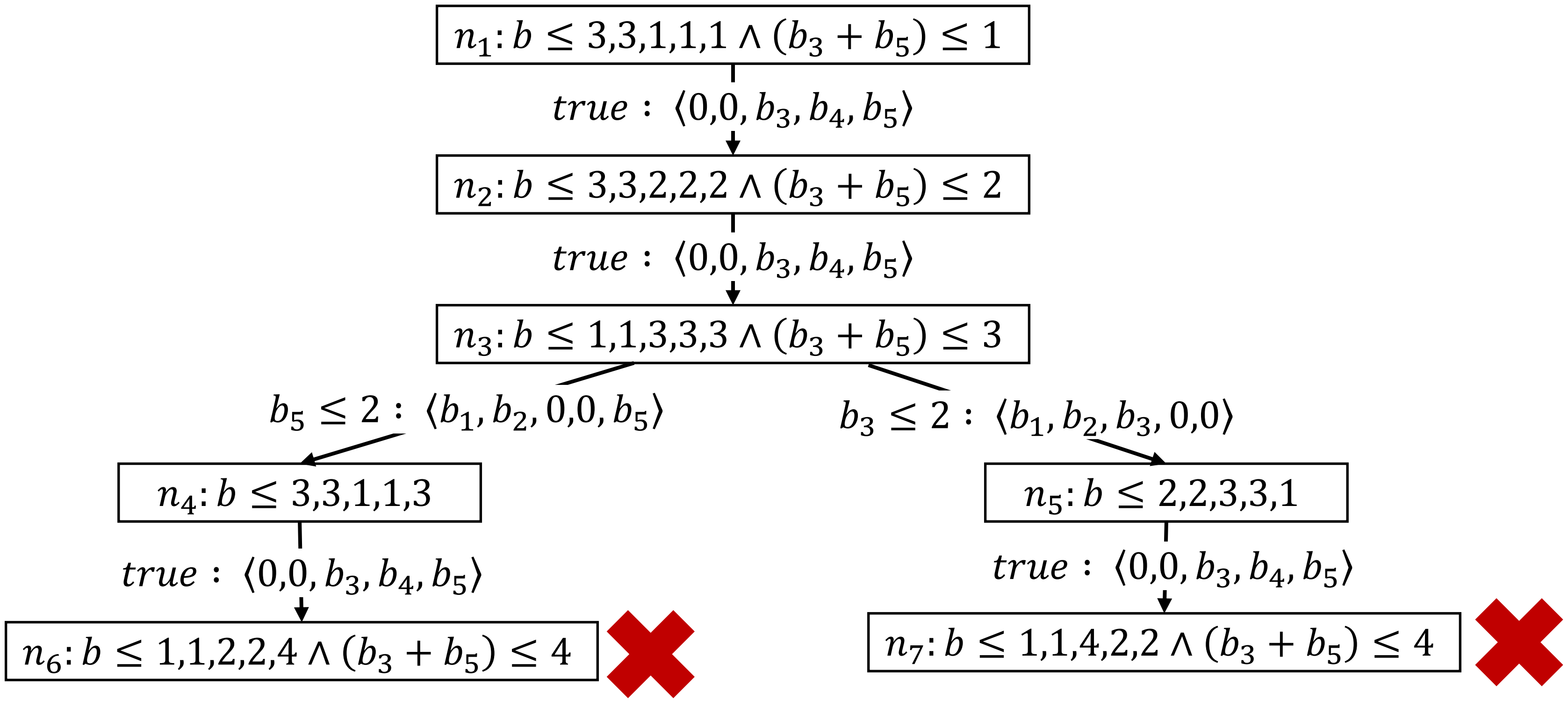
Cinderella's Strategy
Always empty b_1 and b_2

Round 1:

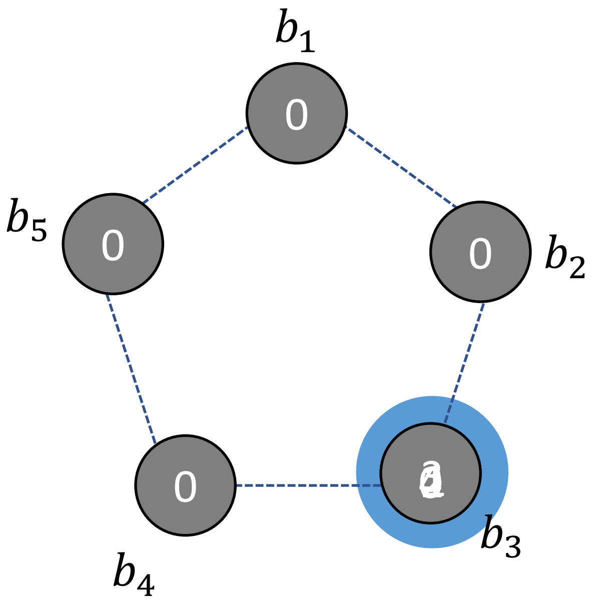
Stepmother fills b_3

[Farzan & Kincaid, 2018]

Refine Safety Tree



Cinderella-Stepmother Game



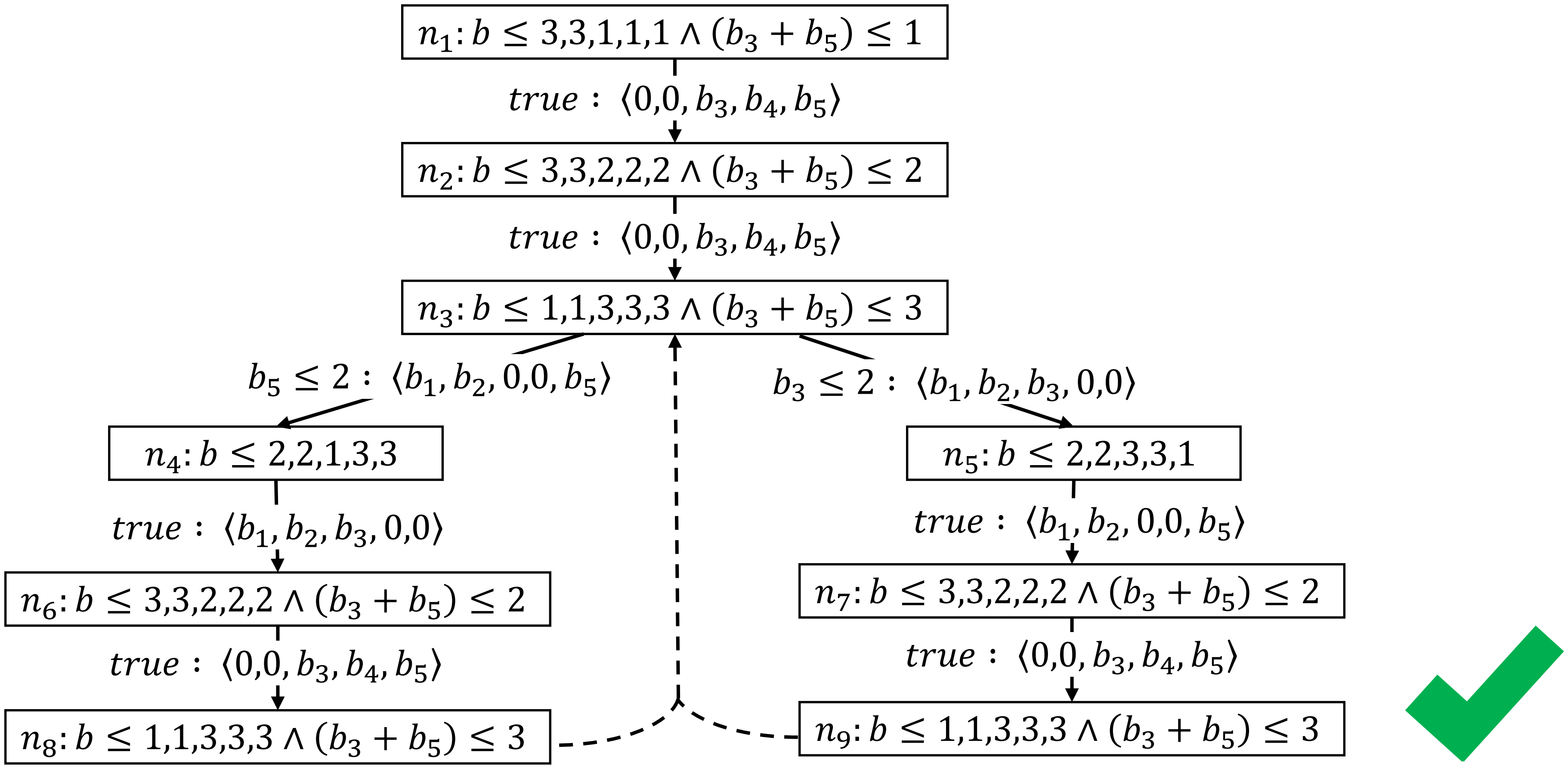
Cinderella's Strategy
Empty b_1 and b_2 always but rd 3
Rd 3 empty b_3 if full else b_5

Round 1:

Stepmother fills b_3

[Farzan & Kincaid, 2018]

Refine Safety Tree



[Farzan & Kincaid, 2018]

Conclusion

- Finite Games
  - Game Semantics
  - Connection between quantification and choices
  - Satisfiability Games
  - Strategy Improvement & Strategy Synthesis
- Infinite Games
  - Reachability Games
  - Strategy Synthesis by generalizing bounded game strategies

References

Farzan, Azadeh and Kincaid, Zachary. Linear Arithmetic Satisfiability via Strategy Improvement. *IJCAI*. 2016.

Farzan, Azadeh and Kincaid, Zachary. Strategy Synthesis for Linear Arithmetic Games. *Proc. ACM Program. Lang*. 2, POPL, Article 61 (January 2018)

Hintikka, Jaakko. 1982. Game-theoretical semantics: insights and prospects. *Notre Dame Journal of Formal Logic Notre-Dame*, Ind. 23, 2 (1982), 219–241.

Khomskii, Yurii. Infinite Games. *Lecture Notes*. University of Sofia, Bulgaria. July 19, 2010. Accessed March 28, 2020. [link]