CS 577 - Divide and Conquer. Applications

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Divide and Conquer

Divide and Conquer (DC)

Overview

- Split problem into smaller sub-problems.
- Solve (usually recurse on) the smaller sub-problems.
- Use the output from the smaller sub-problems to build the solution.

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Tendencies of DC

- Naturally recursive solutions
- Solving complexities often involve recurrences.
- Often used to improve efficiency of efficient solutions, e.g. $O(n^2) \to O(n \log n)$.
- Used in conjunction with other techniques.

Fast Exponentiation

Exponentiation by Squaring

Problem

Compute x^n where x is an integer number and n is a non-negative integer, minimizing the number of multiplications.

Let's assume that multiplication counts per one step*

**Note:* In the real world, as numbers grow larger, the cost of multiplication increases significantly. For example, $2 \times 3 = 6$, but $1,234 \times 56,789 \approx 70$ *million.*

Exponentiation by Squaring

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🤔What is the complexity of the naive method $x^n = \underbrace{x \cdot x \cdots x}_{n \text{times}}$?

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$O(n)$.

Divide & Conquer Approach v1

🤔Discussion: Suggest how to divide the problem.

Divide & Conquer Approach v1

Reducing the problem

Consider the following cases for n:

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🤔How many recursive calls?

Divide & Conquer Approach v1

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🤔How many recursive calls? 1 for each case.

Divide & Conquer Approach v1

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🤔How many recursive calls? 1 for each case. Cost per call? $O(1)$ for each multiplication.

Divide & Conquer Approach v2

Reducing the problem

Consider the following cases for n:

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- If n is odd, $x^n = x \cdot x^{n-1}$.

Example: Fast Exponentiation Algorithm

Divide & Conquer Approach v2

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Consider the following cases for n:

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Example: Fast Exponentiation Algorithm

- Base Case: If $n = 0$, return 1.

Divide & Conquer Approach v2

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- If n is even, $x^n = (x^{n/2})^2$.
- If n is odd, $x^n = x \cdot x^{n-1}$.

Example: Fast Exponentiation Algorithm

- Base Case: If $n = 0$, return 1.
- Recursive Case:
  - If n is even, return $\texttt{fastExp}(x, n/2)^2$.
  - If n is odd, return $x \cdot \texttt{fastExp}(x, n-1)$.

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- Combine: The result is computed as the recursion unwinds.
- Recurrence: $T(n) \leq T(n/2) + O(1) = O(\log n)$.

Integer Multiplication

Integer Multiplication

Partial Product Method:

$$
\begin{array}{r}
12 \\
\times\, 13 \\
\hline
36 \\
12 \\
\hline
156
\end{array}
\qquad\qquad
\begin{array}{r}
1100 \\
\times\, 1101 \\
\hline
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$$

Problem

Multiple two n-length binary numbers x and y, counting every bitwise operation.

Integer Multiplication

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🤔What is the complexity of the partial product method?

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Problem

Multiple two n-length binary numbers x and y, counting every bitwise operation.

What is the complexity of the partial product method? $O(n^2)$.

Divide & Conquer v1

🤔Discussion : Suggest how to divide the problem.

Divide & Conquer v1

High and low bits

Consider $x = x_1 \cdot 2^{n/2} + x_0$ and $y = y_1 \cdot 2^{n/2} + y_0$.

$$\begin{aligned} xy &= (x_1 \cdot 2^{n/2} + x_0)(y_1 \cdot 2^{n/2} + y_0) \\ &= x_1 y_1 \cdot 2^n + (x_1 y_0 + x_0 y_1) \cdot 2^{n/2} + x_0 y_0 \end{aligned}$$

In decimal system:

$$x = 12 \cdot 10^2 + 34 = 1200 + 34 = 1234$$

$$y = 56 \cdot 10^2 + 78 = 5600 + 78 = 5678$$

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xy &= (x_1 \cdot 2^{n/2} + x_0)(y_1 \cdot 2^{n/2} + y_0) \\
&= x_1 y_1 \cdot 2^n + (x_1 y_0 + x_0 y_1) \cdot 2^{n/2} + x_0 y_0
\end{aligned}
$$

🤔How many recursive calls? 4.

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🤔How many recursive calls? 4. Cost per call?

$$\begin{cases} x_1, y_1 \text{ are the } n/2 \text{ highest digits of } x, y \\ x_0, y_0 \text{ are the } n/2 \text{ lowest digits of } x, y \\ x_1y_1, x_1y_0, x_0y_1, x_0y_0 \text{ are } n\text{-digits numbers} \\ \alpha \cdot 2^k \text{ can be done by shifting } k \text{ digits} \\ \text{Summation of } \ell\text{-digit numbers requires } O(\ell) \text{ bit-operations.} \end{cases}$$

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🤔What is the size of the subproblem in recursive calls?

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$$T(n) \le 4T(n/2) + cn$$

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$$T(n) \leq 4T(n/2) + cn = O\left(n^{\lg 4}\right) = O\left(n^2\right)$$

Divide & Conquer v2

High and low bits

Consider $x = x_1 \cdot 2^{n/2} + x_0$ and $y = y_1 \cdot 2^{n/2} + y_0$.

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Hint: $(x_1 + x_0)(y_1 + y_0) = x_1 y_1 + x_1 y_0 + x_0 y_1 + x_0 y_0$.

Exercise: Design an algorithm with 3 Recursive Calls

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Exercise: Design an algorithm with 3 Recursive Calls

- Recursions:
  - $p := \texttt{intMult}(x_1 + x_0, y_1 + y_0)$
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- Combine: Return $x_1 y_1 \cdot 2^n + (p - x_1 y_1 - x_0 y_0) \cdot 2^{n/2} + x_0 y_0$

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- Recurrence: $T(n) \leq 3T(n/2) + O(n) = O\left(n^{\lg 3}\right) = O\left(n^{1.59}\right)$

Matrix Multiplication

Matrix Multiplication

Problem

Multiple two *nxn* matrices, A and B. Let $C = AB$.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 4 + 2 \cdot 2 & 1 \cdot 3 + 2 \cdot 1 \\ 3 \cdot 4 + 4 \cdot 2 & 3 \cdot 3 + 4 \cdot 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 20 & 13 \end{bmatrix}$$

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Algorithm: Naïve Method

```
for i ← 1 to n do
    for j ← 1 to n do
        for k ← 1 to n do
            C[i][j]+ = A[i][k] · B[k][j]
        end
    end
end
```

What is the complexity of the Naïve Method?

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🤔What is the complexity of the Naïve Method? $O(n^3)$.

Divide & Conquer v1

🤔Discussion !!!: Suggest how to divide the problem.

🤔Our standard D&C questions:

Divide & Conquer v1

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$$\left[\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right]\left[\begin{array}{c|c} e & f \\ \hline g & h \end{array}\right] = \left[\begin{array}{c|c} ae+bg & af+bh \\ \hline ce+dg & cf+dh \end{array}\right]$$

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- 🤔 How many recursive calls?

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- Cost per call?

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- 🤔 What is the recurrence?

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$$T(n) \leq 8T(n/2) + cn^2$$

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- What is the recurrence?

$$T(n) \leq 8T(n/2) + cn^2 = O\left(n^{\lg 8}\right) = O\left(n^3\right)$$

Divide & Conquer v2

$$\left[\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right]\left[\begin{array}{c|c} e & f \\ \hline g & h \end{array}\right] = \left[\begin{array}{c|c} p_5 + p_4 - p_2 + p_6 & p_1 + p_2 \\ \hline p_3 + p_4 & p_1 + p_5 - p_3 - p_7 \end{array}\right]$$

Strassen's Method (1969)

- $p_1 := a(f - h)$
- $p_2 := (a + b)h$
- $p_3 := (c + d)e$
- $p_4 := d(g - e)$
- $p_5 := (a + d)(e + h)$
- $p_6 := (b - d)(g + h)$
- $p_7 := (a - c)(e + f)$

Divide & Conquer v2

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- $p_6 := (b - d)(g + h)$
- $p_7 := (a - c)(e + f)$

🤔 What is the recurrence?

Divide & Conquer v2

$$\left[\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right]\left[\begin{array}{c|c} e & f \\ \hline g & h \end{array}\right] = \left[\begin{array}{c|c} p_5 + p_4 - p_2 + p_6 & p_1 + p_2 \\ \hline p_3 + p_4 & p_1 + p_5 - p_3 - p_7 \end{array}\right]$$

Strassen's Method (1969)

- $p_1 := a(f - h)$
- $p_2 := (a + b)h$
- $p_3 := (c + d)e$
- $p_4 := d(g - e)$
- $p_5 := (a + d)(e + h)$
- $p_6 := (b - d)(g + h)$
- $p_7 := (a - c)(e + f)$

What is the recurrence?

$$T(n) \leq 7T(n/2) + cn^2 = O\left(n^{\lg 7}\right) = O\left(n^{2.8074}\right)$$

Divide & Conquer v2

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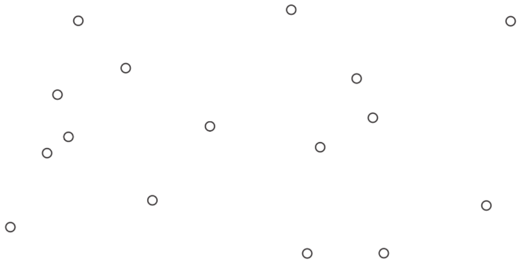
Current Champ: $O(n^{2.373})$

Virginia Vassilevska Williams, MIT

Closest Pairs

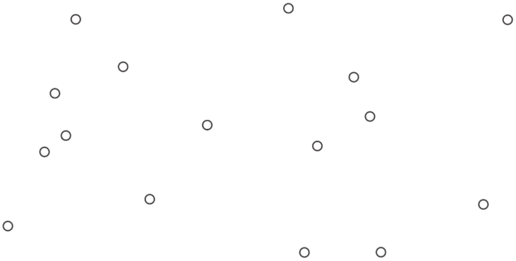
Finding the Closes Pair of Points



Problem

Given a set of n points, $\mathcal{P} = \{p_1, p_2, \ldots, p_n\}$, in the plane. Find the closest pair. That is, solve $\arg\min_{(p_i, p_j) \in \mathcal{P}} \{d(p_i, p_j)\}$, where $d(\cdot, \cdot)$ is the Euclidean distance.

Finding the Closes Pair of Points



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What is the $O(n^2)$ solution?

1-d Version

1-d Closest Pair

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The points are on the line.

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$O(n \log n)$ for 1-d Closest Pair

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$O(n \log n)$ for 1-d Closest Pair

- Sort the points

1-d Version

1-d Closest Pair

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$O(n \log n)$ for 1-d Closest Pair

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1-d Version

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The points are on the line.

$O(n \log n)$ for 1-d Closest Pair

- Sort the points ($O(n \log n)$).
- Walk through sorted points and find minimum pair

1-d Version

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The points are on the line.

$O(n \log n)$ for 1-d Closest Pair

- Sort the points ($O(n \log n)$).
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2-d Closest Pair

Divide and Conquer

1. Divide: Split point set (in half?).

2-d Closest Pair

Divide and Conquer

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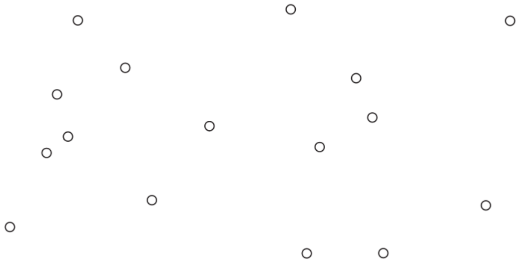
2. Conquer: Find closest pair in each partition.

2-d Closest Pair

Divide and Conquer

1. Divide: Split point set (in half?).
2. Conquer: Find closest pair in each partition.
3. Combine: Merge the solutions.

1. Divide: Split the Points



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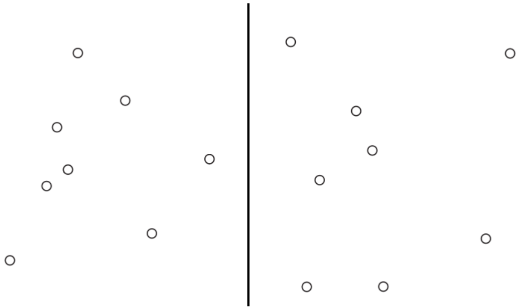

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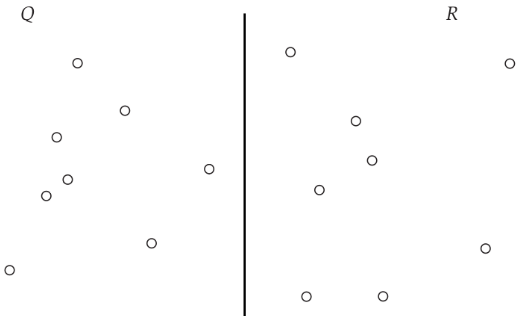

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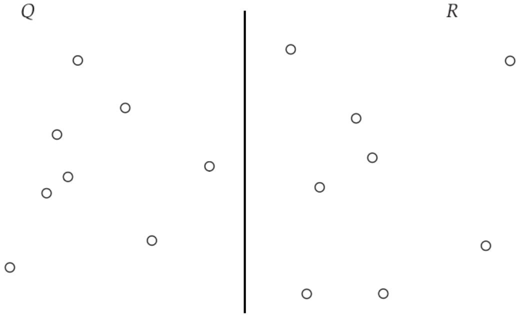
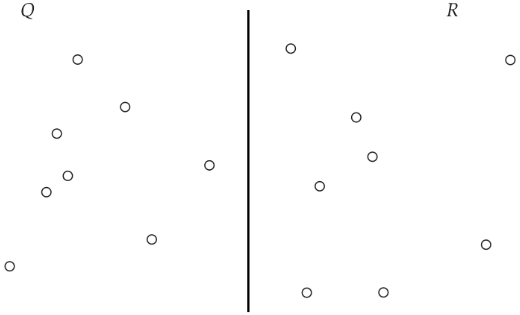


Definitions

- $\mathcal{P}_x$: Points sorted by x-coordinate.
- $\mathcal{P}_y$: Points sorted by y-coordinate.
- Q (resp. R) is left (resp. right) half of $\mathcal{P}_x$.

2. Conquer: Find the min in Q and R



Key Observations

- From $\mathcal{P}_x$ and $\mathcal{P}_y$: We can create Q_x, Q_y, R_x, R_y without resorting.

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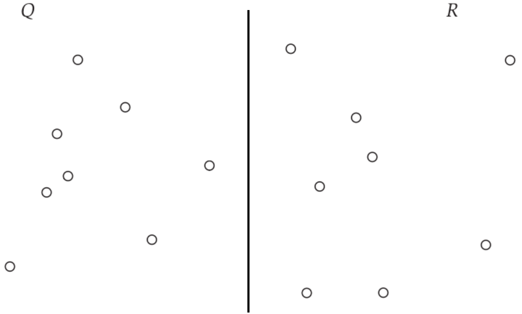

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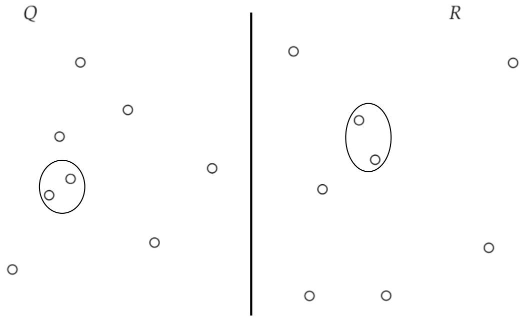
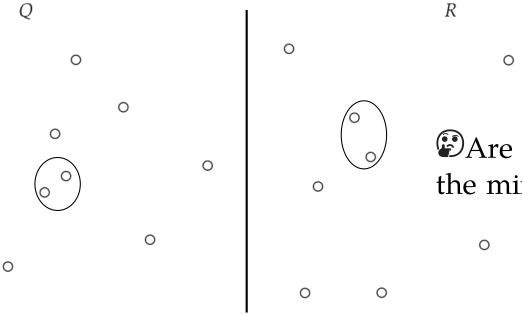


Key Observations

- From $\mathcal{P}_x$ and $\mathcal{P}_y$: We can create Q_x, Q_y, R_x, R_y without resorting.
- Running time for this: $O(n)$.
- Let (q_0^*, q_1^*) and (r_0^*, r_1^*) be closest pairs in Q and R.

3. Combine the Solutions.



🤔Are one of these always the minimum of $\mathcal{P}$?

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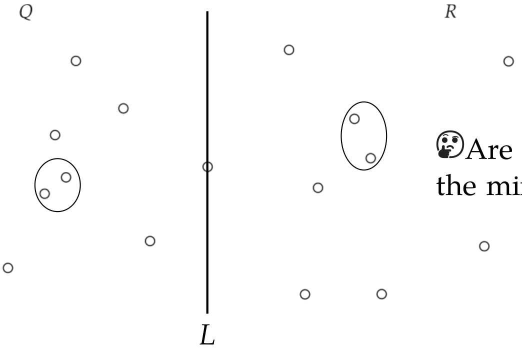
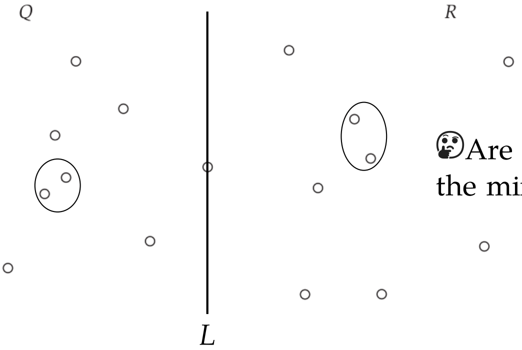


🤔Are one of these always the minimum of $\mathcal{P}$?

Claim 1

🤔 *Let* $\delta := \min\{d(q_0^*, q_1^*), d(r_0^*, r_1^*)\}$*. If there exists a* $q \in Q$ *and an* $r \in R$ *for which* $d(q, r) < \delta$*, then each of q and r are within* □ *of L.*

3. Combine the Solutions.



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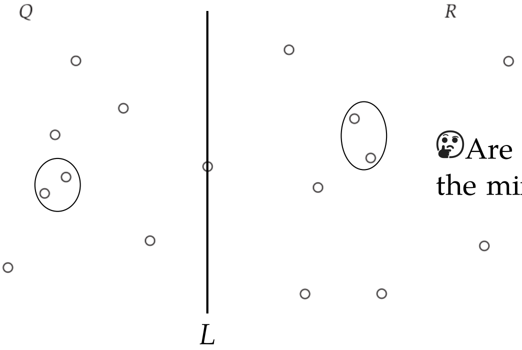

🤔Are one of these always the minimum of $\mathcal{P}$?

Lemma 1

Let S be the set of points within δ of L. If there exists a $s, s' \in S$ and $d(s, s') < \delta$, then s and s' are within 15 positions of each other in S_y.

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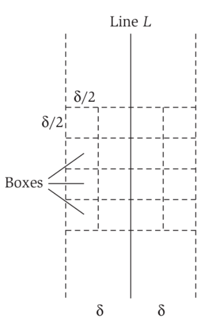

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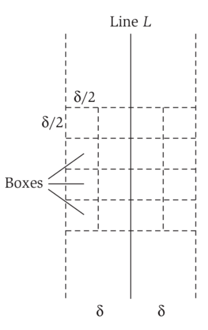

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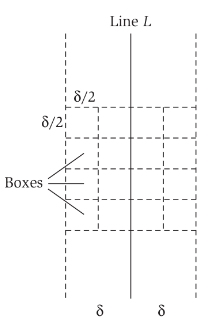

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- At most 1 point per square else contradicts definition of δ.

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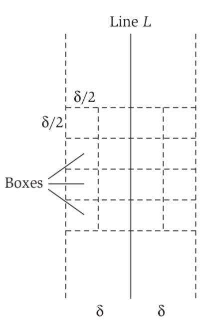

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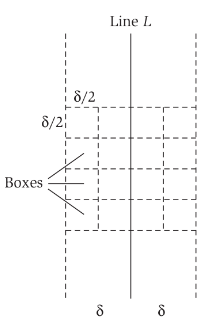

- Partition δ-space around L into $\delta/2$ squares.
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- By counting argument, s and s' are separated by 3 rows which is at least $3\delta/2$. □

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Completing the Algorithm

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Completing the Algorithm

- Find the min pair (s, s') in S.

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Completing the Algorithm

- Find the min pair (s, s') in S.
  - For each $p \in S$, check the distance to each of next 15 points in S_y.

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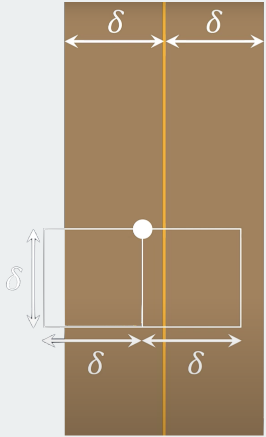
Completing the Algorithm

- Find the min pair (s, s') in S.
    - For each $p \in S$, check the distance to each of next 15 points in S_y.
- If $d(s, s') < \delta$, return (s, s')
- else return min of (q_0^*, q_1^*) and (r_0^*, r_1^*).

Is 15 elements optimal?

Geometry Fun Question



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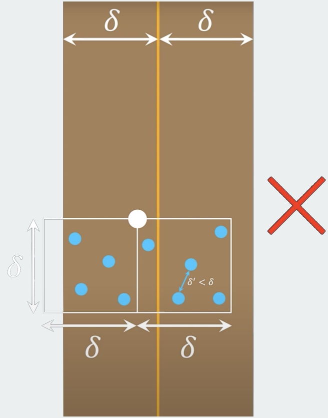

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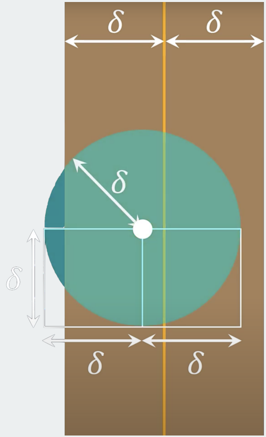

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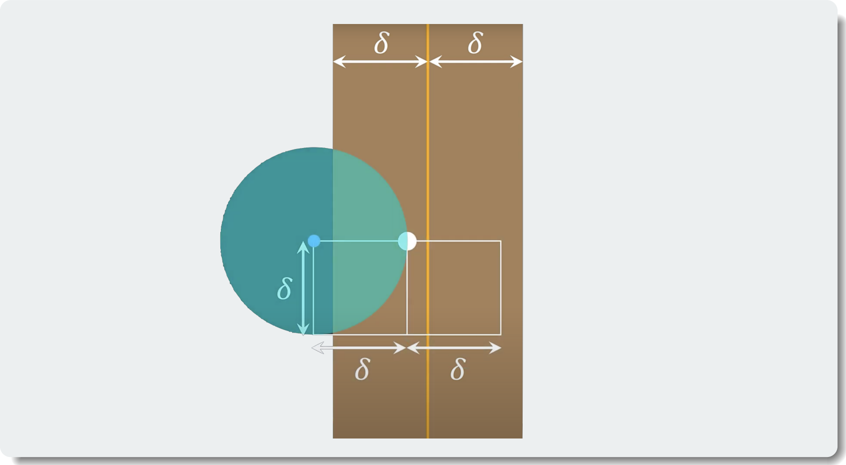

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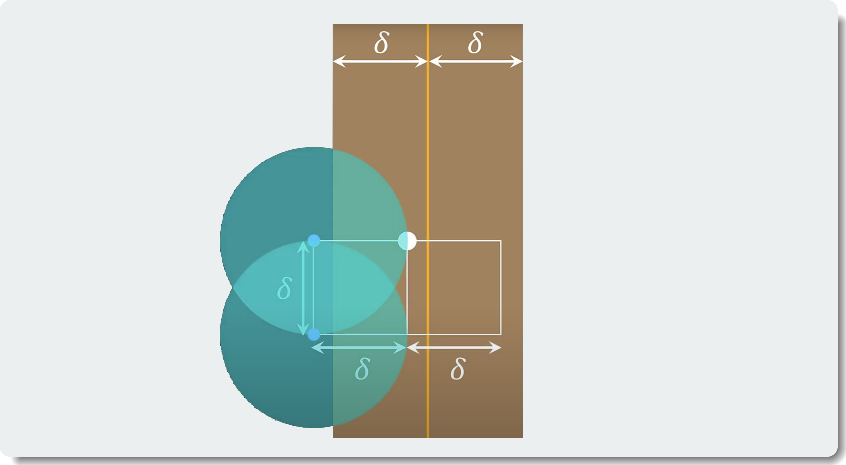

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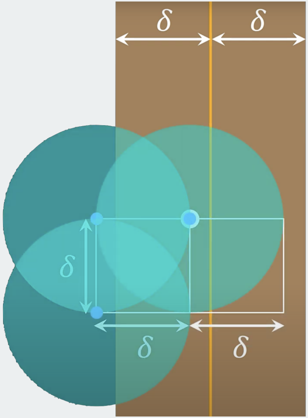

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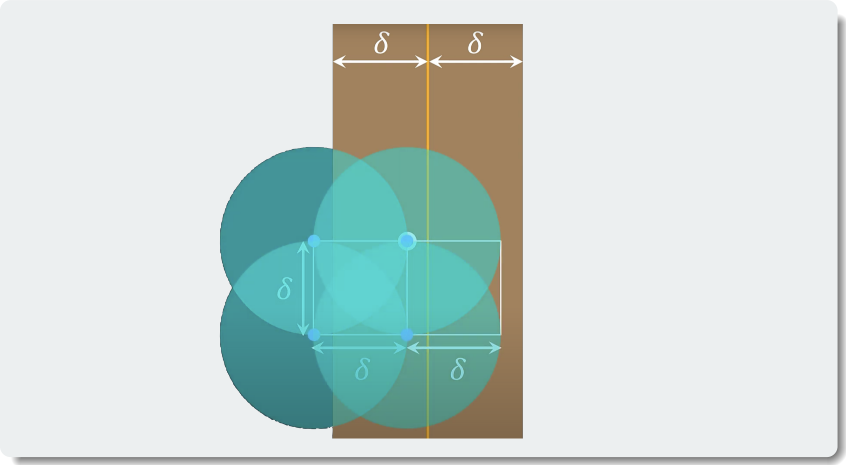

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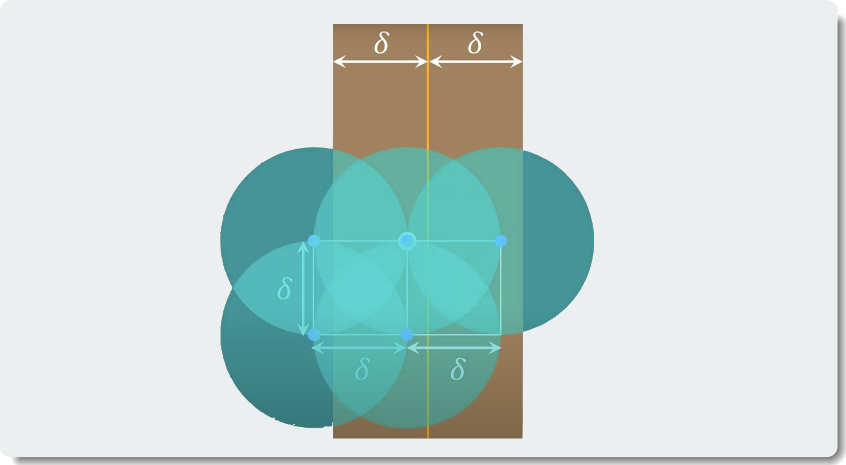

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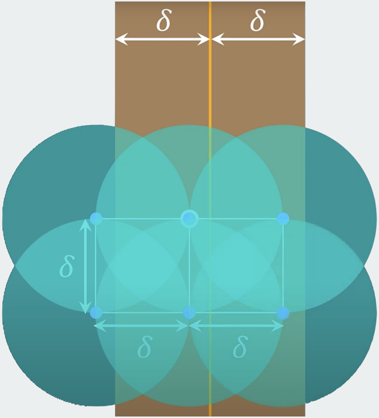

Completing the Analysis

Correctness of the Algorithm

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- By induction on the number of points.
- Use the definition of the algorithm and the claims establish in Step 3.

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- Sorting by x and by y ($O(n \log n)$).
- 🤔 How many recursive calls?

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- Sorting by x and by y ($O(n \log n)$).
- How many recursive calls? 2.

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- 🤔 What is the size of the recursive calls?

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- Sorting by x and by y ($O(n \log n)$).
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- Sorting by x and by y ($O(n \log n)$).
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- What is the recurrence?

$$T(n) \leq 2T(n/2) + cn = O(n \log n) \; .$$

Closest Pair Problem (Divide & Conquer)

```
def dist(p1, p2):
    return ((p1[0] - p2[0])**2 + (p1[1] - p2[1])**2)**0.5

def brute_force(P):
    return min([dist(P[i], P[j]) for i in range(len(P)) for j in range(i + 1, len(P))], default=float('inf'))

def closest_split_pair(Px, Py, delta, best_pair):
    middle = Px[len(Px) // 2][0]
    S = [p for p in Py if middle - delta <= p[0] <= middle + delta]
    best = delta
    for i in range(len(S) - 1):
        min_dist, j_best = min((dist(S[i], S[j]), j) for j in range(i + 1, min(i + 7, len(S))))
        best, best_pair = (min_dist, (i, j_best)) if min_dist <= best else (best, best_pair)
    return best, best_pair
```

Closest Pair Problem (Divide & Conquer)

```python
def closest_pair_rec(Px, Py):
    if len(Px) <= 3:
        return brute_force(Px)

    mid = len(Px) // 2
    Qx = Px[:mid]
    Rx = Px[mid:]

    midpoint = Px[mid][0]
    Qy = [point for point in Py if point[0] <= midpoint]
    Ry = [point for point in Py if point[0] > midpoint]

    (d1, pair1) = closest_pair_rec(Qx, Qy)
    (d2, pair2) = closest_pair_rec(Rx, Ry)
    d, best_pair = (d1, pair1) if d1 <= d2 else (d2, pair2)
    (d3, pair3) = closest_split_pair(Px, Py, d, best_pair)

    return (d, best_pair) if d <= d3 else (d3, pair3)

def closest_pair(points):
    Px = sorted(points, key=lambda x: x[0])
    Py = sorted(points, key=lambda x: x[1])
    return closest_pair_rec(Px, Py)
```

Max Subarray

Max Subarray

Problem

Given an array A of integers, find the (non-empty) contiguous subarray of A of maximum sum.

Max Subarray

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Given an array A of integers, find the (non-empty) contiguous subarray of A of maximum sum.

Exercise – Teams of 3 or so

- Solve the problem in $\Theta(n^2)$.
- Solve the problem in $O(n \log n)$.
- Prove correctness and complexity.

Part 1: Give a $\Theta(n^2)$ solution.

Algorithm: CheckAllSubarrays

Input : Array A of n ints.
Output: Max subarray in A.

```
Let M be an empty array
for i := 1 to len(A) do
    for j := i to len(A) do
        if sum(A[i..j]) > sum(M) then
            M := A[i..j]
        end
    end
end
return M
```

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- Correct: Checks all possible contiguous subarrays.

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        end
    end
end
return M

Analysis

- Correct: Checks all possible contiguous subarrays.
- Complexity:
    - Re-calculating the sum will make it $O(n^3)$. Key is to calculate the sum as you iterate.
    - For each i, check $n - i + 1$ ends. Overall:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \Theta(n^2) .$$

Part 2: Give an $O(n \log n)$ solution.

Algorithm: MaxSubarray

Input : Array A of n ints.
Output: Max subarray in A.
if $|A| = 1$ **then return** $A[1]$
A_1 := MaxSubarray(Front-half of A)
A_2 := MaxSubarray(Back-half of A)
M :=MidMaxSubarray(A)
return Array with max sum of $\{A_1, A_2, M\}$

PART 2: GIVE AN $O(n \log n)$ SOLUTION.

Algorithm: MAXSUBARRAY

Input : Array A of n ints.
Output: Max subarray in A.

if $|A| = 1$ **then return** $A[1]$
A_1 := MAXSUBARRAY(Front-half of A)
A_2 := MAXSUBARRAY(Back-half of A)
M :=MIDMAXSUBARRAY(A)
return Array with max sum of $\{A_1, A_2, M\}$

Algorithm: MIDMAXSUBARRAY

Input : Array A of n ints.
Output: Max subarray that crosses midpoint A.

m := mid-point of A
L := max subarray in $A[i, m-1]$ for $i = m-1 \to 1$
R := max subarray in $A[m, j]$ for $j = m \to n$
return $L \cup R$ `// subarray formed by combining` L `and` R.

Part 2: Give an $O(n \log n)$ solution.

Algorithm: MaxSubarray

Input : Array A of n ints.
Output: Max subarray in A.
if $|A| = 1$ **then return** $A[1]$
A_1 := MaxSubarray(Front-half of A)
A_2 := MaxSubarray(Back-half of A)
M :=MidMaxSubarray(A)
return Array with max sum of $\{A_1, A_2, M\}$

Analysis

- Correctness: By induction, A_1 and A_2 are max for subarray and M is max mid-crossing array.
- Complexity: Same recurrence as MergeSort.

Appendix

References

Image Sources I

https://brand.wisc.edu/web/logos/

https://people.csail.mit.edu/virgi/