CS 577 - Dynamic Programming

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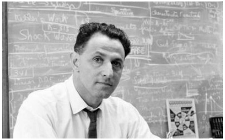
Dynamic Programming

DYNAMIC PROGRAMMING



Richard Bellman

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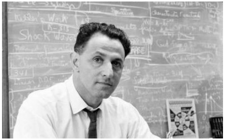
Dynamic Programming



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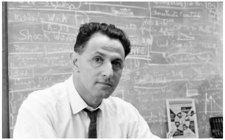
Why "Dynamic Programming"?

Reasons for the name:

- In the 1950s, "programming" was about "planning" rather than coding.
- "Dynamic" is exciting – Air Force director didn't like research and wanted pizzazz.
- "Dynamic" sounds better than "linear" (Re: rival Dantzig).

Dynamic Programming



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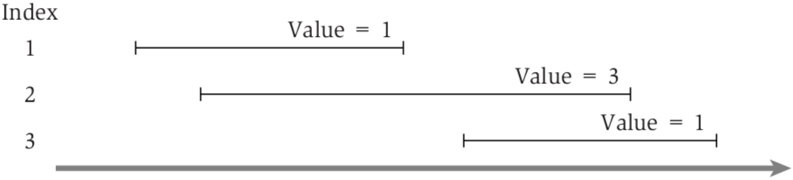
What is it?

- Your new favourite algorithmic technique.
- Extreme Divide and Conquer
- Many sub-problems, but not quite brute-force.
- Dynamic in that it calculates a bunch of solutions from the "smallest" to the "largest".

Weighted Interval Scheduling

WEIGHTED INTERVAL SCHEDULING



Problem Definition

- Requests: $\sigma = \{r_1, \cdots, r_n\}$

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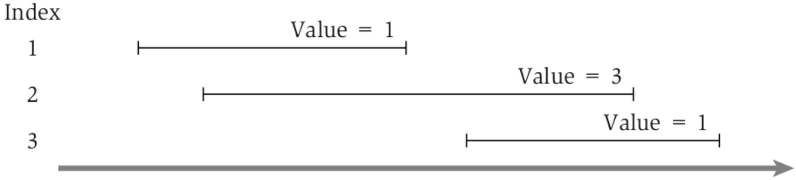

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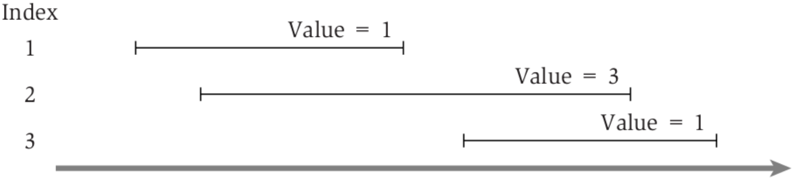

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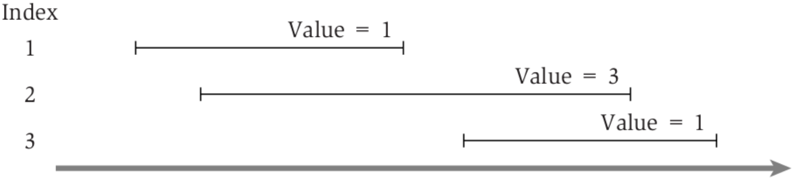

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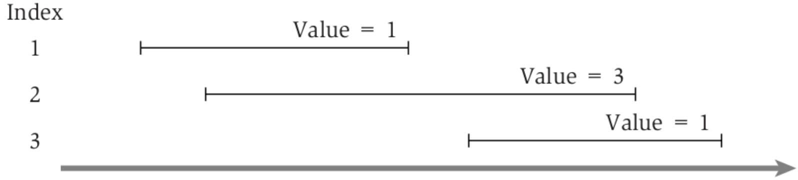

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What is the value of the FF heuristic?

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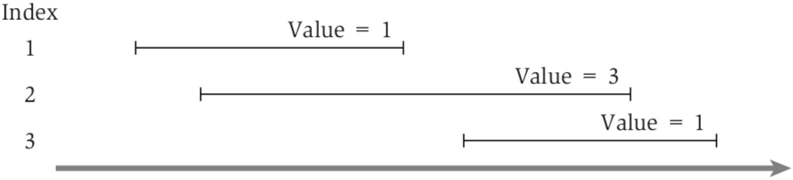

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What is the value of the FF heuristic? 2.

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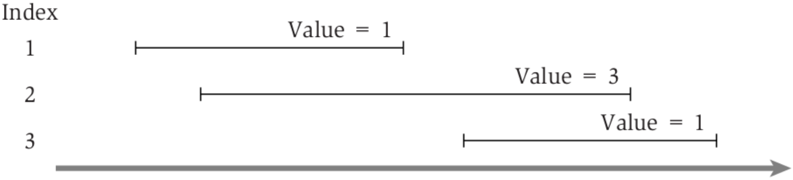

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🤔What is the value of the FF heuristic? 2.

🤔What is the optimal value?

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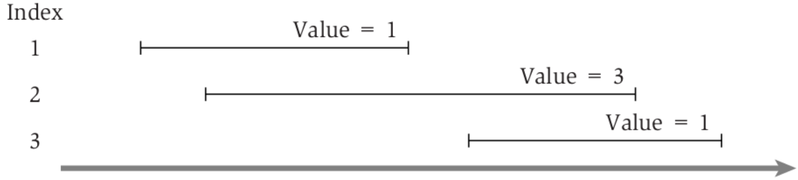

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🤔What is the optimal value? 3.

Recursive Solution

Recursive Procedure

1. Assume σ ordered by finish time (asc).

Proof of optimality.

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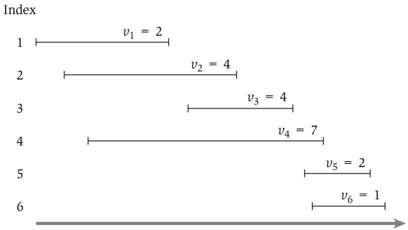
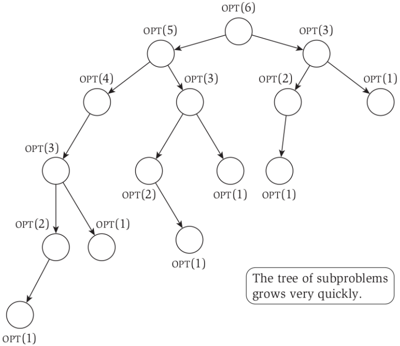
Base cases: $j = 0$ or $j = 1$: Only 1 possible optimal solution.
Inductive step:

- By ind hyp, we have opt for $j - 1$ and opt for i.
- Sorting order assures the dichotomy that the last interval is either in the solution or not.
- Take the max of whether or not a given interval is included. $\square$

CONSIDER THE RECURSION

$$\textsc{opt}(j) = \max(\textsc{opt}(j-1), \textsc{opt}(i) + v_j)$$





Consider the Recursion

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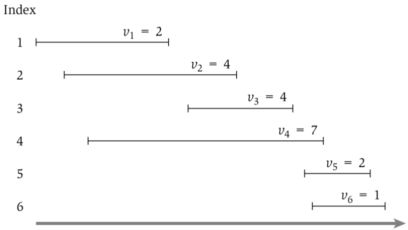

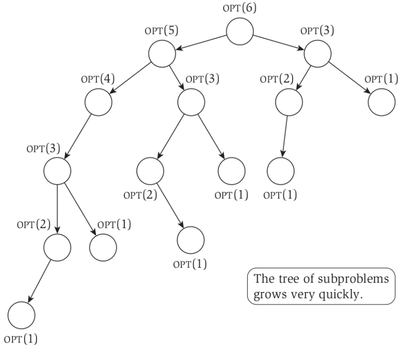

🤔What is the asymptotic number of recursive calls with n jobs?

Consider the Recursion

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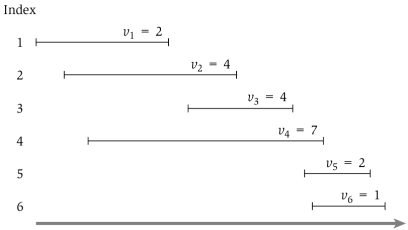

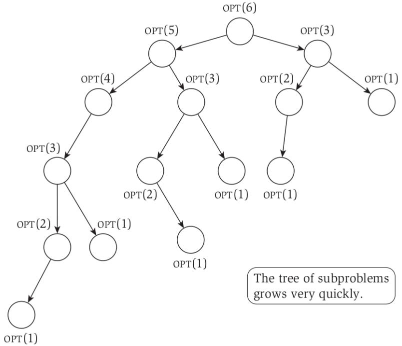

🤔What is the asymptotic number of recursive calls with n jobs?
$O(2^n)$

Memoizing the Recursion

Memoization

- Not a typo.
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Basic Technique

- Calculate once: store the value in array and retrieve for future calls.
- Can be implemented recursively, but tends to be more natural as an iterative process.

Dynamic Program Solution

Algorithm: WeightIntDP

Sort σ by finish time
$m[0] := 0$
for $j = 1$ to n **do**
    Find index i
    $m[j] = \max(m[j-1], m[i] + v_j)$
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DP Solutions

- DP algorithms are formulaic.
- We understand how loops work.
- NO Pseudocode.

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We want:

- Definitions required for algorithm to work
- Description of matrix
- Bellman Equation
- Location of solution, order to populate the matrix

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- 1D array M, where $M[j]$ is the maximum value of a compatible schedule for the first j items in sorted σ. Initialize $M[1] = v_1$.

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Solution, order to populate

- The maximum value of a compatible schedule for the n jobs is found at $M[n]$. Populate from 2 to n.

Analyze the Algorithm

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- Preprocessing:

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- Preprocessing:
  - Sorting jobs: $O(n \log n)$.

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- Populate the matrix:

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- Preprocessing:
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Overall:

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Overall: $O(n^2)$ linear search, $O(n \log n)$ binary search

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What about the schedule S?

ANALYZE THE ALGORITHM

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What about the schedule S?

Trace back from the optimal value:

- Job j is part of the optimal schedule from 1 to j iff $v_j + \textsc{opt}(i) \geq \textsc{opt}(j-1)$

Basic DP Outline

Algorithm Template

- Preprocessing of data
- Populate the matrix:
  - Iterate over the cells in the correct order.
  - Understand the work done per cell.

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2. The solution to the larger problem can be efficiently calculated from the subproblems.

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Algorithm Guidelines

1. There are only a polynomial number of subproblems.
2. The solution to the larger problem can be efficiently calculated from the subproblems.
3. Natural ordering of the subproblems from "smallest" to "largest".

Longest Increasing Subsequence

Longest Increasing Subsequence

Problem

- Given an integer array $A[1..n]$.
- Find the longest increasing subsequence. That is, let i be a sequence of indexes, we have $A[i_k] < A[i_{k+1}]$ for all k.

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🤔For an array of length n, how many subsequences? 2^n

Recursive Approach

Algorithm: LIS

Input : Integer k, and array of integers $A[1..n]$.
Output: Return length of LIS where every value $> k$.
Exo: Complete the algorithm

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else if $A[1] \leq k$ **then**
  return LIS$(k, A[2..n])$
else
  skip :=LIS$(k, A[2..n])$
  take :=LIS$(A[1], A[2..n]) + 1$
  return $\max\{skip, take\}$
end

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    return max{skip, take}
end
```

🤔For an array $A[1..n]$, how would you find the length of the LIS using the LIS($\cdot$) algorithm?

RECURSIVE APPROACH

Algorithm: LIS

Input : Integer k, and array of integers $A[1..n]$.
Output: Return length of LIS where every value $> k$.
if $n = 0$ **then** **return** 0
else if $A[1] \le k$ **then**
  return $\text{LIS}(k, A[2..n])$
else
  $skip$:=$\text{LIS}(k, A[2..n])$
  $take$:=$\text{LIS}(A[1], A[2..n]) + 1$
  return $\max\{skip, take\}$
end

🤔For an array $A[1..n]$, how would you find the length of the LIS using the LIS($\cdot$) algorithm? $\text{LIS}(-\infty, A[1..n])$

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🤔Run time of the algorithm for a length n array?

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🤔Run time of the algorithm for a length n array? $O(2^n)$

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🤔Run time of the algorithm for a length n array? $O(2^n)$

🤔How many distinct recursive calls for a length n array?

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🤔Run time of the algorithm for a length n array? $O(2^n)$
🤔How many distinct recursive calls for a length n array? $O(n^2)$

Dynamic Program for LIS

Description of matrix

🤔Number of dimensions of array?

Dynamic Program for LIS

Description of matrix

🤔Number of dimensions of array? 2

Dynamic Program for LIS

Description of matrix

2D array L, where $L[i,j]$ is the maximum LIS of $A[j..n]$ with every item $> A[i], i < j$.

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Solution and populating L

- Solution in $L[0][1]$; add $A[0] = -\infty$.
- Populate j from n to 1; i from 0 to $j-1$ or $j-1$ to 0.

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- 🤔Run time:

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- Run time: $O(n^2)$

DYNAMIC PROGRAMMING FOR GAMES

Dynamic Programming for Games

Games

- Some number of players (1 to many).
- Set of rules with some objective.
- Huge domain, started by Von Neumann, that spans many fields such as Economics, Math, Biology, and Computer Science.

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DP for Games

In many games, DP is a natural paradigm for an optimal strategy.

Coins in a Line

Players

Two players:

Alice
(Player A)

Bob
(Player B)

Coins in a Line

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- n (even) coins in a line; each coin has a value.
- Starting with Alice, each player will pick a coin from the head or the tail.

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- Winner: Player with the max value at the end; winning player keeps the coins.

Greedy Approaches

Largest Coin

🤔Give a counter-example.

GREEDY APPROACHES

Largest Coin

```
[1,3,6,3]
A: 3; [1,3,6]
B: 6; [1,3]
A: 6; [1]
B: 7; []
```

Greedy Approaches

Largest Coin

Even or Odd

```
[1,3,6,3,1,3]
A: 3; [1,3,6,3,1]
B: 1; [1,3,6,3]
A: 6; [1,3,6]
B: 7; [1,3]
A: 9; [1]
B: 8; []
```

Greedy Approaches

Largest Coin

Even or Odd

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A: 3; [1,3,6,3,1]
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- Alice can always win.

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- Alice can always win.
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A: 9; [1]
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```

```
[1,3,6,3,1,3]
A: 3; [1,3,6,3,1]
B: 1; [3,6,3,1]
A: 4; [3,6,3]
B: 4; [6,3]
A: 10; [3]
B: 7; []
```

- Alice can always win.
- But are we optimal? No

NATURAL DICHOTOMY

🤔What is the natural dichotomy?

Natural Dichotomy

Head or Tail?

- Two players: Assume that Bob will play optimally.

Natural Dichotomy

Head or Tail?

- Two players: Assume that Bob will play optimally.
- For Alice's kth turn:
  - Coin array: $C[i..j]$
  - $\max\{c[i] + \text{BobOpt}(c[i+1..j]), c[j] + \text{BobOpt}(c[i..j-1])\}$

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🤔How many dimensions for DP array?

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🤔How many dimensions for DP array? 2

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DP Description

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    - $M[i,j]$ is the maximum value possible for Alice when choosing from $c[i..j]$, assuming Bob plays optimally.

Head or Tail DP

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- Populate i from $n-2$ to 1; j from n to 3 for $i < j-1$.

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- Runtime:

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- $M[i, i] = c[i]$ for all i.
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- Populate i from $n - 2$ to 1; j from n to 3 for $i < j - 1$.
- Solution: $M[1, n]$
- Runtime: $O(n^2)$

HEAD OR TAIL DP

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- Proof of correctness:

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- Solution: $M[1,n]$
- Runtime: $O(n^2)$
- Proof of correctness: Strong induction on the cell population order.

Max Subarray

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Problem

Given an array A of integers, find the (non-empty) contiguous subarray of A of maximum sum.

Max Subarray

Problem

Given an array A of integers, find the (non-empty) contiguous subarray of A of maximum sum.

Exercise – Teams of 3 or so

- Solve the problem in $\Theta(n^2)$.
- Solve the problem in $O(n \log n)$.
- Prove correctness and complexity.

PART 1: GIVE A $\Theta(n^2)$ SOLUTION.

Algorithm: CHECKALLSUBARRAYS

Input : Array A of n ints.
Output: Max subarray in A.

```
Let M be an empty array
for i := 1 to len(A) do
    for j := i to len(A) do
        if sum(A[i..j]) > sum(M) then
            M := A[i..j]
        end
    end
end
return M
```

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Analysis

- Correct: Checks all possible contiguous subarrays.

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return M
```

Analysis

- Correct: Checks all possible contiguous subarrays.
- Complexity:
  - Re-calculating the sum will make it $O(n^3)$. Key is to calculate the sum as you iterate.
  - For each i, check $n - i + 1$ ends. Overall:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \Theta(n^2) \; .$$

Part 2: Give an $O(n \log n)$ solution.

Algorithm: MaxSubarray

Input : Array A of n ints.
Output: Max subarray in A.
if $|A| = 1$ **then return** $A[1]$
A_1 := MaxSubarray(Front-half of A)
A_2 := MaxSubarray(Back-half of A)
M :=MidMaxSubarray(A)
return Array with max sum of $\{A_1, A_2, M\}$

PART 2: GIVE AN $O(n \log n)$ SOLUTION.

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$A_2 :=$ MAXSUBARRAY(Back-half of A)
$M :=$MIDMAXSUBARRAY(A)
return $\underline{\text{Array with max sum of } \{A_1, A_2, M\}}$

Algorithm: MIDMAXSUBARRAY

Input : Array A of n ints.
Output: Max subarray that crosses midpoint A.
$m :=$ mid-point of A
$L :=$ max subarray in $A[i, m-1]$ for $i = m-1 \to 1$
$R :=$ max subarray in $A[m, j]$ for $j = m \to n$
return $\underline{L \cup R}$ `// subarray formed by combining` L `and` R.

PART 2: GIVE AN $O(n \log n)$ SOLUTION.

Algorithm: MAXSUBARRAY

Input : Array A of n ints.
Output: Max subarray in A.
if $|A| = 1$ **then return** $A[1]$
A_1 := MAXSUBARRAY(Front-half of A)
A_2 := MAXSUBARRAY(Back-half of A)
M :=MIDMAXSUBARRAY(A)
return Array with max sum of $\{A_1, A_2, M\}$

Analysis

- Correctness: By induction, A_1 and A_2 are max for subarray and M is max mid-crossing array.
- Complexity: Same recurrence as MERGESORT.

Max Subarray

Problem

Given an array A of integers, find the (non-empty) contiguous subarray of A of maximum sum.

Exercise – Teams of 3 or so

- Solve the problem in $\Theta(n^2)$.
- Solve the problem in $O(n \log n)$.
- Prove correctness and complexity.
- **With dynamic programming, solve the problem in $O(n)$!**

Part 3: Give an $O(n)$ solution.

DP Solution

- 1D array s, where $s[i]$ contains the value of the max subarray ending at i. ($O(n)$ cells)
- Bellman equation: $s[i] = \max(s[i-1] + A[i], A[i])$. ($O(1)$ time)
- Solutions is: $\max_j\{s[j]\}$. ($O(n)$ time)

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But we need the subarray not the value!

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But we need the subarray not the value!

- Use a parallel array that memoizes the starting index of the subarray ending at i:

$$\text{start}[i] = \begin{cases} \text{start}[i-1] & \text{if } s[i-1] + a[i] > a[i] \\ i & \text{, otherwise} \end{cases}$$

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- Use a parallel array that memoizes the starting index of the subarray ending at i:

$$\text{start}[i] = \begin{cases} \text{start}[i-1] & \text{if } s[i-1] + a[i] > a[i] \\ i & \text{, otherwise} \end{cases}$$

- Or, trace back from max value at index j until $s[i] = A[i]$.

Subset and Knapsack

Subset Problem

Problem Definition

- A single machine that we can use for time W.

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Greedy Heuristics

- Decreasing weights:

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Greedy Heuristics

- Decreasing weights: $\{\mathrm{W}/2 + 1, \mathrm{W}/2, \mathrm{W}/2\}$

Subset Problem

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- Decreasing weights: $\{W/2 + 1, W/2, W/2\}$
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1D Approach

- if $n \notin S$, then $v[n] = v[n-1]$

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2D Approach

Dynamic Programming Approach

2D Approach

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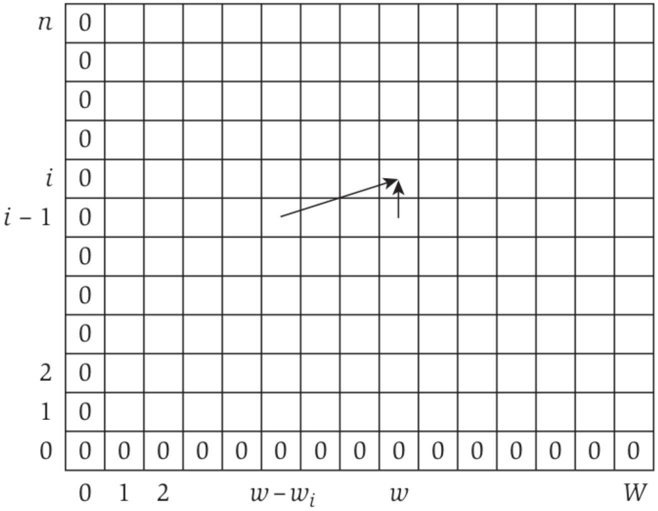
🤔Running time to populate the matrix: $O(nW)$

🤔Is this polynomial? No, pseudo-polynomial because of W which is unbounded.

SUBSET VISUALIZATION

Matrix Visualization:



SUBSET VISUALIZATION

Example Run:

$$W = 6, \text{ items } w_1 = 2,\ w_2 = 2,\ w_3 = 3$$

3							
2							
1							
0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6

Initial values

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3							
2	0	0	2	2	4	4	4
1	0	0	2	2	2	2	2
0	0	0	0	0	0	0	0
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Filling in values for $i = 2$

Subset Visualization

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3							
2							
1							
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Initial values

3							
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1	0	0	2	2	2	2	2
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Filling in values for *i* = 1

3							
2	0	0	2	2	4	4	4
1	0	0	2	2	2	2	2
0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6

Filling in values for *i* = 2

3	0	0	2	3	4	5	5
2	0	0	2	2	4	4	4
1	0	0	2	2	2	2	2
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Filling in values for *i* = 3

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How can we recover the subset itself?
🤔Running time of recovery of subset: $O(n)$

KNAPSACK EXTENSION

Problem Definition

- You are a thief with a knapsack that can carry W weight of goods.

Knapsack Extension

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- What is the subset S of items to steal that maximizes $\sum_{i \in S} v_i$ with the constraint that $\sum_{i \in S} w_i \leq W$?

Exercise: Solve this with DP in $O(nW)$.

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- 2D Matrix:
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Minimum number of letter

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- deletions: removing a letter,
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to change string $A[1..m]$ to string $B[1..n]$.

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Ex: TUESDAY → THUESDAY → THURSDAY

Or, align and count mismatched letters

```
T UESDAY
THURSDAY
```

RECURSIVE APPROACH

Smaller Subproblems

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Dynamic Program for Edit Distance

Description of matrix

Number of dimensions of array?

Dynamic Program for Edit Distance

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🤔Number of dimensions of array? 2

Dynamic Program for Edit Distance

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2D array E, where $E[i, j]$ is the edit distance for $A[1..i]$ and $B[1..j]$.

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Solution and populating L

- Solution in
- Set $E[0,j] = j$; $E[i,0] = i$; populate from 1 to n, 1 to m.

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Space Savings

Bellman Equation

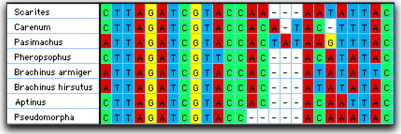
$$E[i,j] = \begin{cases} i, \text{ if } j = 0 \\ j, \text{ if } i = 0 \\ \min\{E[i,j-1]+1, E[i-1,j]+1, \\ \qquad E[i-1,j-1]+A[i] \neq B[j]\}, \text{ otherwise} \end{cases}$$

How much space do we need?

- Notice that $E[i][j]$ depends on $E[i,j-1]$, $E[i-1,j]$, and $E[i-1,j-1]$.
- We only need previous and current row of matrix for calculations.

Sequence Alignment

SEQUENCE ALIGNMENT



Scarites	C	T	T	A	G	A	T	C	G	T	A	C	C	A	A	-	-	-	A	A	T	A	T	T	A	C
Carenum	C	T	T	A	G	A	T	C	G	T	A	C	C	A	C	A	-	T	A	C	-	T	T	T	A	C
Pasimachus	A	T	T	A	G	A	T	C	G	T	A	C	C	A	C	T	A	T	A	A	G	T	T	T	A	C
Pheropsophus	C	T	T	A	G	A	T	C	G	T	T	C	C	A	C	-	-	-	A	C	A	T	A	T	A	C
Brachinus armiger	A	T	T	A	G	A	T	C	G	T	A	C	C	A	C	-	-	-	A	T	A	T	A	T	T	C
Brachinus hirsutus	A	T	T	A	G	A	T	C	G	T	A	C	C	A	C	-	-	-	A	T	A	T	A	T	A	C
Aptinus	C	T	T	A	G	A	T	C	G	T	A	C	C	A	C	-	-	-	A	C	A	A	T	T	A	C
Pseudomorpha	C	T	T	A	G	A	T	C	G	T	A	C	C	-	-	-	-	-	A	C	A	A	A	T	A	C

Needleman–Wunsch Problem

- An alphabet S.
- Strings $X = x_1x_2 \ldots x_m$ and $Y = y_1y_2 \ldots y_n$ from S.
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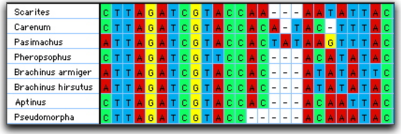

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Scarites	CTTAGATCGTACCAA---AATATTAC
Carenum	CTTAGATCGTACCACA-TAC-TTTAC
Pasimachus	ATTAGATCGTACCACTATAAGTTTAC
Pheropsophus	CTTAGATCGTTCCAC---ACATATAC
Brachinus armiger	ATTAGATCGTACCAC---ATATATTC
Brachinus hirsutus	ATTAGATCGTACCAC---ATATATAC
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Pseudomorpha	CTTAGATCGTACC-----ACAAATAC

$\delta = 3$; $\alpha_{pp} = 0$; $\alpha_{pq} = 1$

6: What is the cost of the matching:

```
o-currance
occurrence
```

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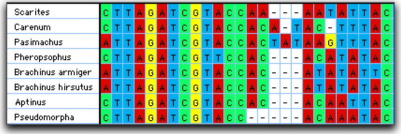

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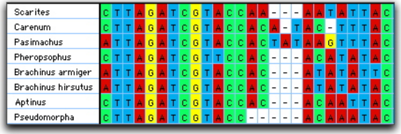

$\delta = 1; \alpha_{pp} = 0; \alpha_{pq} = 4$

8: What is the cost of the matching:

`o-currance`

`occurrence`

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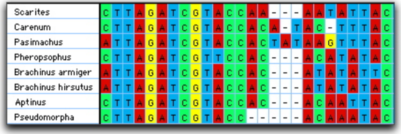

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```
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Designing Needleman–Wunsch Algorithm

Basic Dichotomy

In optimal alignment M, either $(m, n) \in M$ or $(m, n) \notin M$.

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- Contradicts the non-crossing requirement.

$\square$

Designing Needleman–Wunsch Algorithm

Key Concepts for Optimality

In an optimal alignment M, at least one of the following is true:

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- 🤔How many dimensions for the matrix?

Designing Needleman–Wunsch Algorithm

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- 🤔Build the Bellman equation.

Designing Needleman–Wunsch Algorithm

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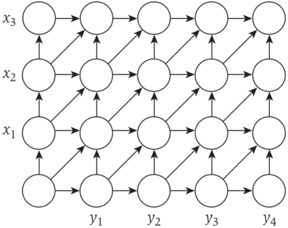
- Runtime: $O(mn)$.

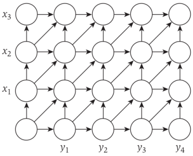
Graphing the Algorithm



Theorem 2

Let $f(i,j)$ denote the minimum cost of a path from $(0,0)$ to (i,j) in G_{XY}. Then, $\forall i,j\, f(i,j) = A[i][j]$.

Graphing the Algorithm



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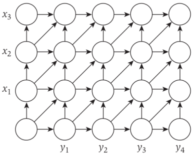

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- By strong induction on

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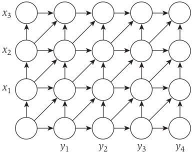

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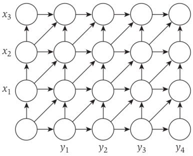

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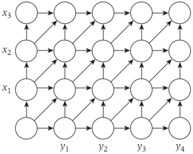

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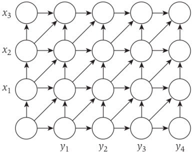

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- Inductive step:

$$\begin{aligned} f(i,j) &= \min\{\alpha_{x_i y_j} + f(i-1,j-1), \delta + f(i-1,j), \delta + f(i,j-1)\} \\ &= \min\{\alpha_{x_i y_j} + A[i-1][j-1], \delta + A[i-1][j], \delta + A[i][j-1]\} \\ &= A[i,j] \end{aligned}$$

$\square$

SEQUENCE ALIGNMENT EXAMPLE

$$A[i][j] = \min\{\alpha_{x_i y_j} + A[i-1][j-1], \delta + A[i-1][j], \delta + A[i][j-1]\}$$

- "mean" vs "name"
- $\delta = 2; \alpha = \begin{cases} 0 & \text{if same letter} \\ 3 & \text{if vowel to consonant} \\ 1 & \text{otherwise} \end{cases}$

n					
a					
e					
m					
-					
	-	n	a	m	e

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n	8	6	5	4	6
a	6	5	3	5	5
e	4	3	2	4	4
m	2	1	3	4	6
-	0	2	4	6	8
	-	n	a	m	e

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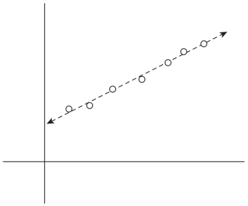
Least Squares

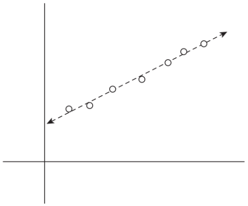
SEGMENTED LEAST SQUARES



Problem Setup

- Set of n points: $P := \{(x_1, y1), (x_2, y_2), \ldots, (x_n, y_n)\}$ on the plane.
- Suppose $x_1 < x_2 < \cdots < x_n$.
- Find $L : y = ax + b$ that minimizes: $\mathrm{Error}(L, P) = \sum_{i=1}^{n} (y_i - ax_i - b)^2$.

Segmented Least Squares



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Problem Formulation

- Partition the points (by x) into contiguous subsets.
- Minimize the sum of $\text{Error}(L, p_i) + C$ for all subsets, where C is a fixed cost per subset.

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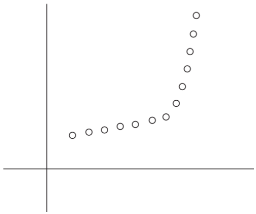

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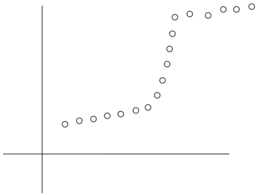

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DP SOLUTION

$$s[j] = \min_{1 \le i \le j} (e_{i,j} + C + s[i-1])$$

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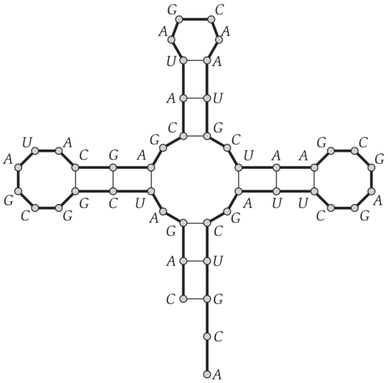
- Overall: $O(n^2)$.

RNA Secondary Structure

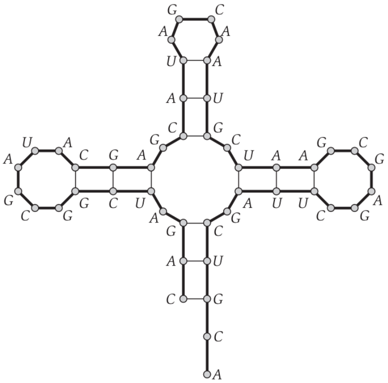
RNA Secondary Structure



Problem Definition

- RNA tends to loop back on itself, forming base pairs.
- RNA alphabet: $\{A, C, G, U\}$.
- Valid pairs: (A, U) or (C, G).

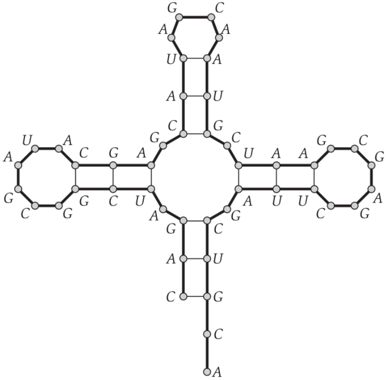
RNA Secondary Structure



Problem Definition

- RNA tends to loop back on itself, forming base pairs.
- RNA alphabet: $\{A, C, G, U\}$.
- Valid pairs: (A, U) or (C, G).
- Input: n length string: $B = b_1 b_2 \dots b_n$
- Output: Determine a secondary structure with maximum number of base pairs.

RNA Secondary Structure



Secondary Structure

$S = \{(i,j)\}$, where $i < j$ and $i,j \in \{1,\ldots,n\}$, such that:

1. No Sharp turns: $i < j - d$ for some constant d.
2. All pairs are valid.
3. S is a matching: no base appears more than once.
4. Non-crossing: For any $(i,j),(i',j') \in S$, we cannot have $i < i' < j < j'$.

First Dynamic Programming Attempt

1D Approach

- 1D array m, where $m[j]$ is the maximum # of pairs among: $b_1 b_2 \ldots b_j$.

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Dichotomy:

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     2. Max pairs in $[t+1, j-1]$: Restricted to $b_{t+1} b_{t+2} \ldots b_{j-1}$ which current DP does not calculate.

Second Dynamic Programming Attempt

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- 2D array m, where $m[i][j]$ is the maximum # of pairs among: $b_i b_{i+1} \ldots b_j$.

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   - Sub-problems:
     1. Max pairs in $[i, t-1]$: $m[i][t-1]$.

SECOND DYNAMIC PROGRAMMING ATTEMPT

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🤔What is the Bellman equation?

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RNA Secondary Structure Example

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4				
3				
2				
1				
j	6	7	8	9

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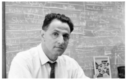
Appendix

References

Image Sources I



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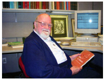

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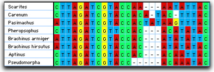

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