CS 577 - Computational Intractability

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Fall 2024

Computational Intractability

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Easy Problems

- Problems that can be solved by efficient algorithms.
- Polynomial running time.
- Complexity class: P

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- Problems for which we do not know how to solve efficiently.

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- Problems for which we do not know how to solve efficiently.
- NP-hard

Computational Intractability

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- Problems that can be solved by efficient algorithms.
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Hard Problems

- Problems for which we do not know how to solve efficiently.
- NP-hard
- NP-complete

Decision Problem

Optimization:

Bipartite Matching

Given a bipartite graph G, find the largest matching.

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Decision:

Bipartite Matching

Given a bipartite graph G, is there a matching of size $\geq k$?

Decision Problem

- binary output: yes / no answer.

Decision Problem

Optimization:

Bipartite Matching

Given a bipartite graph G, find the largest matching.

$\Longleftrightarrow$

Decision:

Bipartite Matching

Given a bipartite graph G, is there a matching of size $\geq k$?

Optimization to Decision

- Solve the optimization version.
- If the solution of size $\geq k$, return yes.

DECISION PROBLEM

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Bipartite Matching

Given a bipartite graph G, find the largest matching.

$\Longleftrightarrow$

Decision:

Bipartite Matching

Given a bipartite graph G, is there a matching of size $\geq k$?

Decision to Optimization

- Upper bound on maximum matching is $N = \min(|A|, |B|)$.
- For $k = N$ to 0, return first k that returns yes.

Decision Problem

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Bipartite Matching

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Decision:

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Decision to Optimization

- Upper bound on maximum matching is $N = \min(|A|, |B|)$.
- For $k = N$ to 0, return first k that returns yes.
  (Or, binary search between $[0, N]$.)

Reductions

Polynomial-Time Reduction

Problem Reduction: $Y \leq_p X$

- Consider any instance of problem Y.
- Assume we have a black-box solver for problem X.

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Y is polynomial-time reducible to X

Suppose $Y \leq_p X$. If X is solvable in polynomial time, then Y can be solved in polynomial time.

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Y is polynomial-time reducible to X

Suppose $Y \leq_p X$. If X is solvable in polynomial time, then Y can be solved in polynomial time.

X is at least as hard as Y

Suppose $Y \leq_p X$. If Y cannot be solved in polynomial time, then X cannot be solved in polynomial time.

Independent Set $\iff$ Vertex Cover

Given a graph G and a number k.

Independent Set (IS)

- Does G contain an IS of size $\geq k$?
- $S \subseteq V$ is independent if no 2 nodes in S are adjacent.

Vertex Cover (VC)

- Does G contain a vertex cover of size $\leq k$?
- $S \subseteq V$ is vertex cover if every edge is incident to at least 1 node in S.

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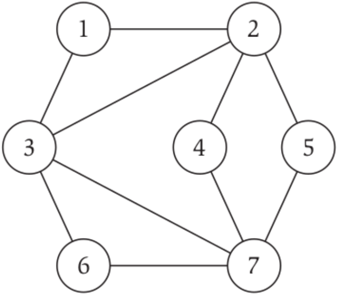

🤔What is size of the largest independent set?

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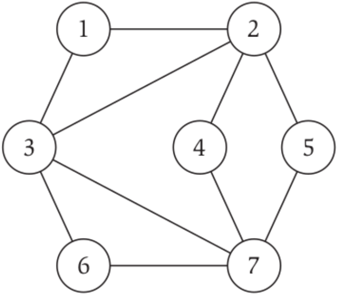

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Theorem 1

Let $G = (V, E)$ be a graph. Then S is an independent set if and only if its complement $V \setminus S$ is a vertex cover.

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Theorem 1

Let $G = (V, E)$ be a graph. Then S is an independent set if and only if its complement $V \setminus S$ is a vertex cover.

Proof.

$\Rightarrow$: Suppose S is an IS. For any edge (u, v), at most one of $\{u, v\} \in S$. Hence, one of $\{u, v\} \in V \setminus S$.

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Given a graph G and a number k.

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Let $G = (V, E)$ be a graph. Then S is an independent set if and only if its complement $V \setminus S$ is a vertex cover.

Proof.

$\Leftarrow$: Suppose $V \setminus S$ is a VC. Any edge (u, v) with both $\in S$ would contradict that $V \setminus S$ is a VC. $\square$

Packing and Covering Problems

Packing Problem

Independent Set

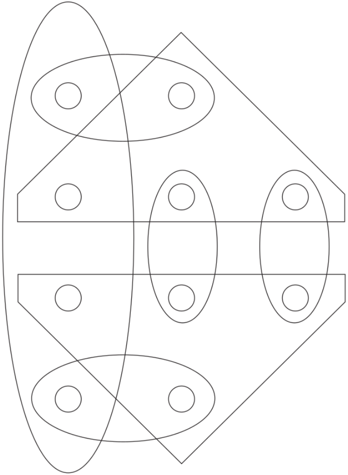
- Goal is to pack as many vertices as possible without violating edge constraints.

Covering Problem

Vertex Cover

- Goal is to cover all the edges in the graph using as few vertices as possible.

Set Cover (SC)



Problem Definition

- A universe U of n elements.
- A collection of subsets of U: $S_1, S_2, \ldots, S_m$.
- A number k.
- Goal: Does there exist a collection of at most k of the subsets whose unions equal U.

Reduction: Vertex Cover (VC) to Set Cover (SC)

Theorem 2

VC $\leq_p$ SC

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Proof.

- 🤔: For the proof, do we assume a VC or a SC black-box?

Reduction: Vertex Cover (VC) to Set Cover (SC)

Theorem 2

VC $\leq_p$ SC

Proof.

- Assume that we have a black-box solver for SC.

REDUCTION: VERTEX COVER (VC) TO SET COVER (SC)

Theorem 2

VC $\leq_p$ SC

Proof.

- Assume that we have a black-box solver for SC.
- Consider an arbitrary instance of VC on $G = (V, E)$.
  - Set $U = E$.
  - For each vertex $v \in V$:
    Create a set consisting of each edge incident to v.

REDUCTION: VERTEX COVER (VC) TO SET COVER (SC)

Theorem 2

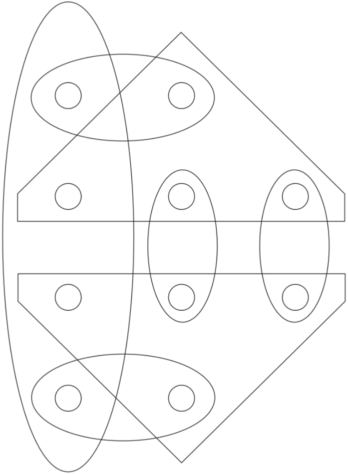
VC $\leq_p$ SC

Proof.

- Assume that we have a black-box solver for SC.
- Consider an arbitrary instance of VC on $G = (V, E)$.
  - Set $U = E$.
  - For each vertex $v \in V$:
    Create a set consisting of each edge incident to v.
- Direct correspondence between VC and SC.
  - VC $\leq k \iff$ SC $\leq k$

□

SET PACKING (SP)



Problem Definition

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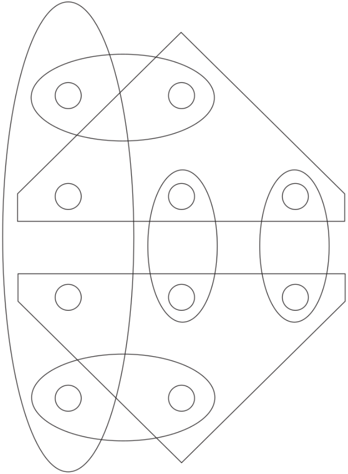
Set Packing (SP)



Exercise: Show that IS $\leq_p$ SP

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Reduction: Independent Set (IS) to Set Packing (SP)

Theorem 3

IS $\leq_p$ SP

REDUCTION: INDEPENDENT SET (IS) TO SET PACKING (SP)

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- Direct correspondence between IS and SP.
  - IS $\geq k \iff$ SP $\geq k$

$\square$

Satisfiability Problem (SAT)

$$(x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee \overline{x_3}) \wedge (x_2 \vee \overline{x_3})$$

Preliminaries

- A set of boolean terms/literals: $X : x_1, \ldots, x_n$.

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- v is a satisfying assignment if $\mathcal{C}$ is 1, i.e., all C_i evaluate to 1.

SATISFIABILITY PROBLEM (SAT)

: What values will satisfy the example?

$$(x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee \overline{x_3}) \wedge (x_2 \vee \overline{x_3})$$

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Three Satifiability (3SAT)

SAT Problem

Given a set of literals: $X : x_1, \ldots, x_n$, and a collection of clauses $\mathcal{C} : C_1 \wedge C_2 \wedge \cdot \wedge C_k$, does there exist a satisfying assignment?

THREE SATIFIABILITY (3SAT)

3SAT Problem

Given a set of literals: $X : x_1, \ldots, x_n$, and a collection of clauses $\mathcal{C} : C_1 \wedge C_2 \wedge \cdot \wedge C_k$, each of length 3, does there exist a satisfying assignment?

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Gadgets

Gadgets are often used to show $Y \leq_p X$.

- A subset of problem X that represents a component of problem Y.

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Gadgets are often used to show $Y \leq_p X$.

- A subset of problem X that represents a component of problem Y.
- A procedure to convert some of the components of Y to a piece of problem X.

3SAT to Independent Set (IS)

Theorem 4

3SAT $\leq_p$ IS

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Proof.

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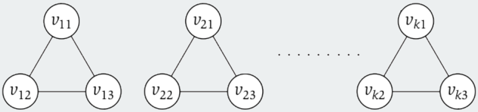

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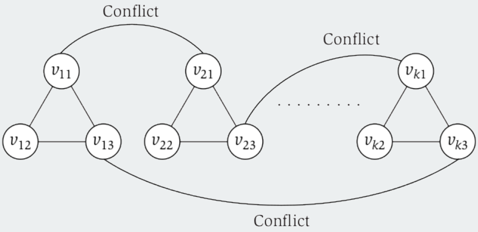

  - Add an edge between $v_{ij} = x_q$ and all $v_{i'j'} = \overline{x_q}$.

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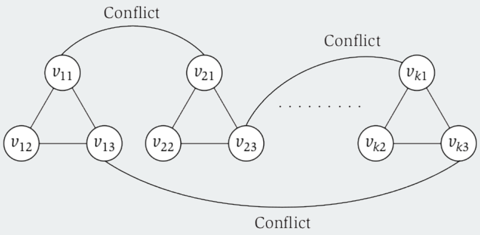

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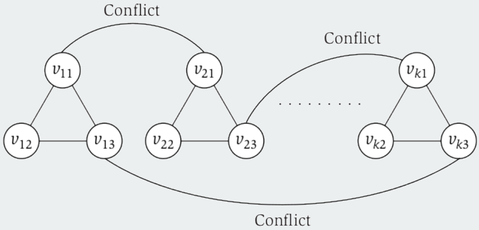

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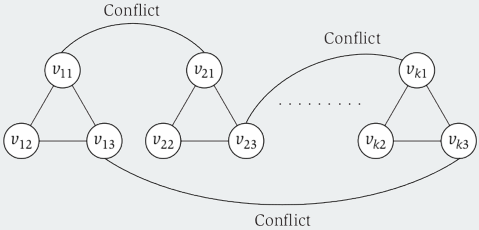

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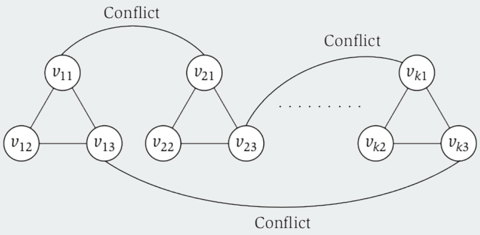

- IS of size $\geq k \iff$ 3SAT is satisfiable.
  - Each node in IS represents a 1 assignment.
  - Within each gadget, only 1 node can be in IS.
  - Conflict edges prevent x_i and $\overline{x_i}$ both being assigned 1.

$\square$

Transitivity of Reductions

Observation 1

If $Z \leq_p Y$, and $Y \leq_p X$, then $Z \leq_p X$.

Transitivity of Reductions

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So,

$$\text{3SAT} \leq_p \text{IS} \leq_p \text{VC} \leq_p \text{SC}$$

and

$$\text{3SAT} \leq_p \text{IS} \leq_p \text{SP}$$

and

$$\text{VC} \leq_p \text{IS} \leq_p \text{SP} .$$

NP

Efficient Certification

Input Formalization

For a problem instance:

- Let s be a binary string that encodes the input.
- $|s|$ is the length of s, i.e., the # of bits in s.

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Polynomial Run-Time

Algorithm A has a polynomial run-time if run-time is $O(\text{poly}(|s|))$ in the worst-case, where $\text{poly}(\cdot)$ is a polynomial function.

Complexity class P

P is the set of all problems for which there exists an algorithm A that solves the problem with polynomial run-time.

Efficient Certification

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Certifier $B(s, t)$ for a problem P:

- s is an input instance of P.
- t is a certificate; a proof that s is a yes-instance.

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- Efficient:
  - For every s, we have $s \in P$ iff there exists a t, $|t| \leq \text{poly}(|s|)$, for which $B(s, t)$ returns yes.

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  - In other words, using t, we can check if s is a yes-instance in polynomial time.

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- $B(s, t)$ returning no does not mean that s is a no-instance

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  - In other words, using t, we can check if s is a yes-instance in polynomial time.
- $B(s, t)$ returning no does not mean that s is a no-instance... only that t is not a valid proof.
- $B(s, t)$ provides a brute-force algorithm: For a given s, check every possible t.

NP PROBLEMS

Complexity Class NP

- **N**on-deterministic, **P**olynomial time: can be solved in polynomial time by testing every t simultaneously (non-deterministic).
- Set of all problems for which there exists an efficient certifier.

NP Problems

Complexity Class NP

- **N**on-deterministic, **P**olynomial time: can be solved in polynomial time by testing every t simultaneously (non-deterministic).
- Set of all problems for which there exists an efficient certifier.

Theorem 5

$P \subseteq NP$

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Theorem 5

P $\subseteq$ NP

Proof.

🤔: Which proof technique?

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- Set of all problems for which there exists an efficient certifier.

Theorem 5

P ⊆ NP

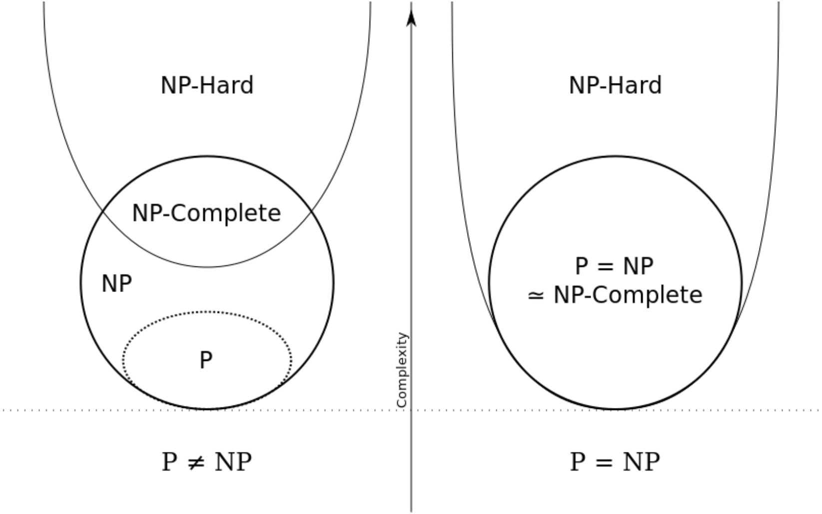
Proof.

- For every $p \in P$, $\exists$ an algorithm A that runs in polynomial time.
- $B(s, t)$ for any t returns $A(s)$.

□

Million Dollar Question: P vs NP

1 of 7 Clay Mathematics Institute Millennium Prize Problems



NP-complete

Hardest NP Problems

NP-Hard

Problem X is NP-Hard if:

- For all $Y \in \text{NP}$, $Y \leq_p X$.
- NP-Hard problem may or may not be in NP.

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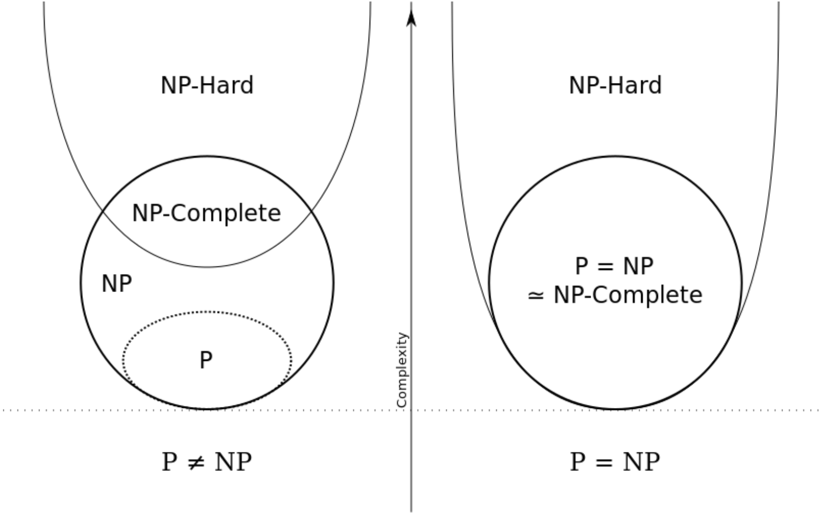

$P \neq NP$ $\qquad$ $P = NP$

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Hardest NP Problems

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Theorem 6

Suppose $X \in$ NP-Complete. Then, X is solvable in polynomial time iff P = NP.

Hardest NP Problems

NP-Complete

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Theorem 6

Suppose $X \in \mathrm{NP}$-Complete. Then, X is solvable in polynomial time iff $\mathrm{P} = \mathrm{NP}$.

Proof.

$\Leftarrow$: Suppose $\mathrm{P} = \mathrm{NP}$, then by definition of P, X can be solved in polynomial time.

Hardest NP Problems

NP-Complete

Problem X is NP-Complete if:

- For all $Y \in \mathrm{NP}$, $Y \leq_p X$.
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Theorem 6

Suppose $X \in \mathrm{NP}$-Complete. Then, X is solvable in polynomial time iff $\mathrm{P} = \mathrm{NP}$.

Proof.

$\Rightarrow$: Suppose X can be solved in polynomial time. Then, by definition of NP-Complete, all problems $\in \mathrm{NP} \leq_p X$. Hence, solvable in polynomial time and $\in P$. $\square$

FIRST NP-COMPLETE PROBLEM

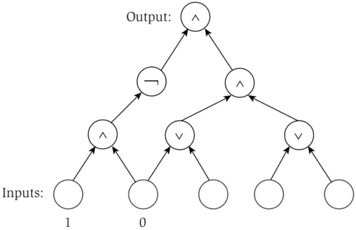
Theorem 6

Cook (1971) – Levin (1973) Theorem [Paraphrase]: Circuit Satisfiability Problem (CSAT) *is* NP-*Complete.*

Stephen Cook (1968)

Leonid Levin (2010)

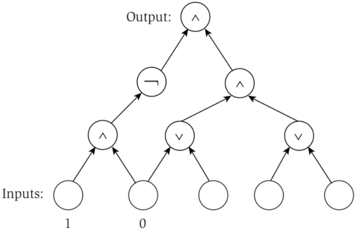
Circuit Satisfiability Problem (CSAT)



Problem Definition

- 3 types of gates: $\wedge$ (AND), $\vee$ (OR), and $\neg$ (NOT).

Circuit Satisfiability Problem (CSAT)



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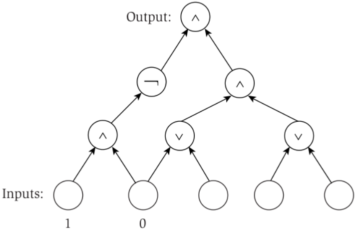

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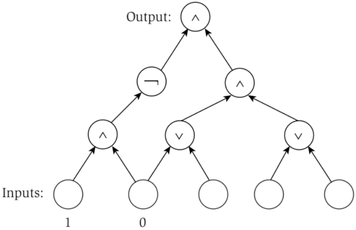

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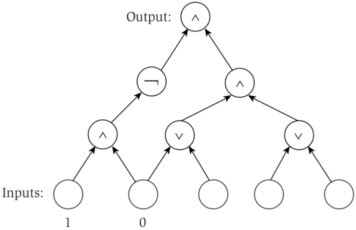

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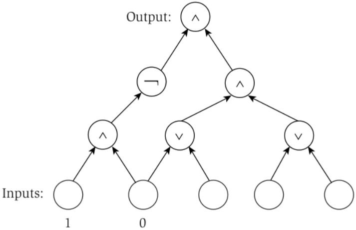

🤔: What is the output with an input of $(1, 0, 0)$?

Problem Definition

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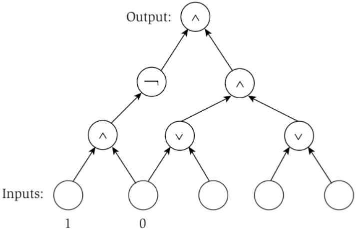

🤔: Give an input that satisfies the example.

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FIRST NP-COMPLETE PROBLEM

Theorem 6

Cook (1971) – Levin (1973) Theorem [Paraphrase]: Circuit Satisfiability Problem (CSAT) *is* NP-*Complete.*

Partial Proof.

1. Show that CSAT $\in NP$:
   - Input size is $\Omega(|V|)$.
   - A single gate can be evaluated in constant time.
   - Evaluate a certificate of the inputs can be verified in $O(|V|)$ time.

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   - We need to show $X \leq_p$ CSAT.
   - By definition for X:
     - X has an input of $|s|$ bits.
     - Produces 1 bit of output (yes/no).
     - $\exists$ an efficient certifier $B_X(\cdot,\cdot)$.

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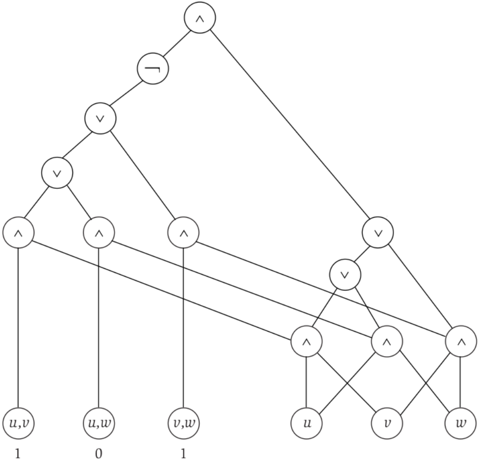
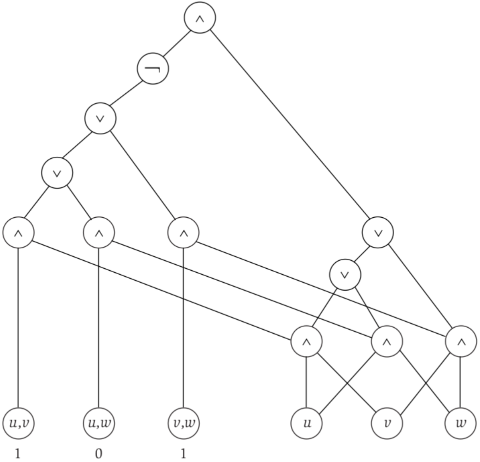
     - The $\text{poly}(n)$ bits are free and used to find a t such that $B_X(s, t)$ is yes.
     - The gates of the circuit are a translation of algorithm B_X.

□

EXAMPLE: INDEPENDENT SET ($k \geq 2$) AS CIRCUIT SATISFIABILITY PROBLEM.



Example: Independent Set ($k \geq 2$) as Circuit Satisfiability Problem.



🤔: Draw the underlying Independent Set graph.

Strategies for Proving NP-Completeness

Showing that Problem X is NP-Complete

Cook Reduction:

1. Prove that $X \in \mathrm{NP}$.
2. Choose a problem $Y \in$ NP-Complete.
3. Prove $Y \leq_p X$.

STRATEGIES FOR PROVING NP-COMPLETENESS

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Typical Step 3

3. Karp Reduction: For an arbitrary instance s_Y of Y, show how to construct, in polynomial time, an instance s_X of X such that s_y is a yes iff s_x is a yes.
   Steps:
   1. Provide efficient reduction.
   2. Prove $\Rightarrow$: if s_Y is a yes, s_X is a yes.
   3. Prove $\Leftarrow$: if s_X is a yes, then s_Y had to have been a yes.

3SAT is NP-Complete

Theorem 7

3SAT *is* NP-*Complete.*

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Exercise: Do step 1.

Show that Problem 3SAT is NP-Complete

Cook Reduction:

1. Prove that 3SAT $\in$ NP.
2. Choose a problem $Y \in$ NP-Complete.
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3SAT is NP-Complete

Theorem 7

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Cook Reduction:

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Proof.

1. Use a truth assignment of the literals as a certificate. This can be verified in polynomial time.

3SAT is NP-Complete

Theorem 7

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Show that Problem 3SAT is NP-Complete

Cook Reduction:

1. Prove that 3SAT $\in$ NP.
2. Choose a problem $Y \in$ NP-Complete.
3. Prove $Y \leq_p$ 3SAT.

Proof.

1. Use a truth assignment of the literals as a certificate. This can be verified in polynomial time.
2. The only NP-Complete problem we know is CSAT.

3SAT is NP-Complete

Proof.

3. For an arbitrary circuit k:

3SAT is NP-Complete

Proof.

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   - Each node v is assigned a variable x_v.

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        - NOT: Let u be the input. We need $x_u = \overline{x_v}$.
          $\rightarrow$ 2 clauses: $(x_v \vee x_u) \wedge (\overline{x_v} \vee \overline{x_u})$.

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     - OR: Let u, w be the inputs. We need $x_v = x_u \vee x_w$.
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     - AND: Let u, w be the inputs. We need $x_v = x_u \wedge x_w$.
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   - For each constant source s:
     $\rightarrow$ 1 clause: (x_s) if 1, and $(\overline{x_s})$ if 0.

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   - For each constant source s:
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   - For the output o: 1 clause (x_o).
   - Convert clauses to length 3:
     - We need 2 variables z_1 and z_2 that are always 0 in a satisfying assignment.
     - To ensure this, we need 4 variables: z_1, z_2, z_3, z_4.

3SAT IS NP-COMPLETE

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   - For the output o: 1 clause (x_o).
   - Convert clauses to length 3:
     - 4 variables: z_1, z_2, z_3, z_4, and 8 clauses for $i \in \{1, 2\}$:
       $(\overline{z_i} \vee z_3 \vee z_4) \wedge (\overline{z_i} \vee \overline{z_3} \vee z_4) \wedge (\overline{z_i} \vee z_3 \vee \overline{z_4}) \wedge (\overline{z_i} \vee \overline{z_3} \vee \overline{z_4})$.

3SAT is NP-Complete

Proof.

3. s_{CSAT} is a yes iff s_{3SAT} is a yes:
   - $\Rightarrow$: If s_{CSAT} is a yes, then the satisfying assignment to the circuit inputs can be used to calculate the value of each gate. By the reduction, these value will satisfy all the clauses of s_{3SAT}.
   - $\Leftarrow$: If s_{3SAT} is a yes, then the assignment of the variables give the satisfying assignment of the circuit inputs, and the reduction guarantees that the assigned values for the nodes match the gate calculations.

$\square$

3SAT is NP-Complete

From our previous reductions

$$3SAT \leq_p IS \leq_p VC \leq_p SC$$

and

$$3SAT \leq_p IS \leq_p SP$$

and the fact that 3SAT is NP-Complete:

Corollary 7

The following problems are NP*-Complete:*

$$3SAT, IS, VC, SC, SP \ .$$

Taxonomy of NP-completeness

Sequencing Problems

Travelling Salesperson Problem (TSP)

- A salesperson must visit n cities $v_1, v_2, \ldots, v_n$.
- Starting at some v_1, visit all cities and return to v_1.
- Distance function: $d(\cdot, \cdot)$ for all pairs of cities (not necessarily symmetric nor metric).
- Optimization: What is the shortest tour?

SEQUENCING PROBLEMS

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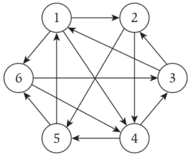
- Decision: Is there a tour of length D?

SEQUENCING PROBLEMS



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Hamiltonian Cycle

- Graph analogue of TSP.
- Hamiltonian cycle: a tour of the nodes of G that visits each node once.
- Given a digraph G, does it contain a Hamiltonian cycle?

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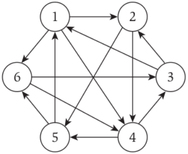

: Does this graph contain a Hamiltonian cycle?

Hamiltonian Cycle

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- Hamiltonian cycle: a tour of the nodes of G that visits each node once.
- Given a digraph G, does it contain a Hamiltonian cycle?

3SAT $\leq_p$ Hamiltonian

Theorem 8

Hamiltonian Cycle is NP*-complete.*

Proof.

1. In NP: A certificate would be a sequence of vertices which can be verified in polynomial time.
2. Choose an NP-complete problem: 3SAT.

3SAT $\leq_p$ Hamiltonian

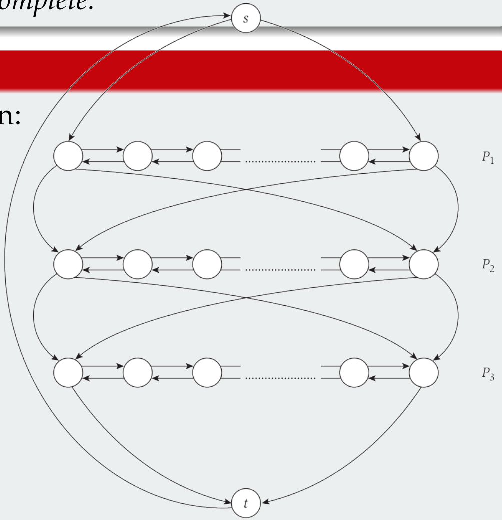
Theorem 8

Hamiltonian Cycle is NP-*complete.*

Proof.

3. 3SAT $\leq_p$ Hamiltonian:

- P_i (containing $3k+2$ nodes) for each X_i: left traversal for 1 and right traversal for 0.



3SAT $\le_p$ Hamiltonian

Theorem 8

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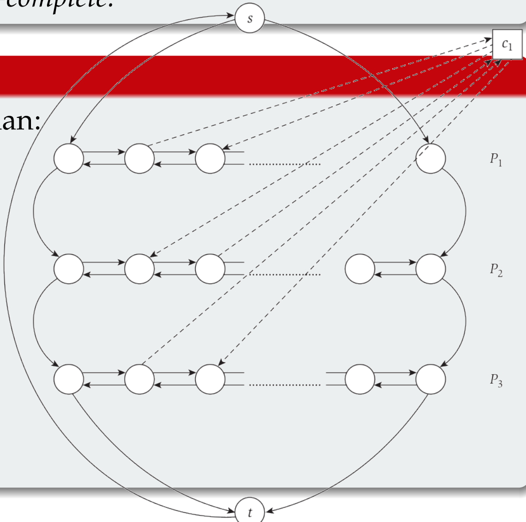
Proof.

3. 3SAT $\le_p$ Hamiltonian:

- C_i for each clause i: Connect based on x_i or $\overline{x_i}$.



3SAT $\leq_p$ Hamiltonian

Theorem 8

Hamiltonian Cycle is NP*-complete.*

Proof.

3. s_{3SAT} is a yes iff $s_{\text{Hamiltonian}}$ is a yes:
   - $\Rightarrow$: If s_{3SAT} is a yes, then each clause node can be visited from one of the paths corresponding to one of the variables when the path is traversed in the direction of the satisfying assignment.
   - $\Leftarrow$: If $s_{\text{Hamiltonian}}$ is a yes, then every clause node is visited, and the direction of each path traversal gives a value assignment for the corresponding variable in s_{3SAT}. The reduction guarantees that value assignment for a variable the path used to traverse the clause node will be the assignment of a variable that satisfies the corresponding clause.

$\square$

Travelling Salesperson

Theorem 9

Travelling Salesperson (TSP) *is* NP-*complete.*

Proof.

Travelling Salesperson

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1. In NP: Certificate that is a tour of the cities.

Travelling Salesperson

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1. In NP: Certificate that is a tour of the cities.
2. 🤔Which *NP*-complete problem?

Travelling Salesperson

Theorem 9

Travelling Salesperson (TSP) *is* NP-*complete.*

Proof.

1. In NP: Certificate that is a tour of the cities.
2. Use Hamiltonian Cycle.
3. Hamiltonian Cycle $\leq_p$ TSP:
   Exo: Try to come up with the reduction.

Travelling Salesperson

Theorem 9

Travelling Salesperson (TSP) *is* NP-*complete.*

Proof.

1. In NP: Certificate that is a tour of the cities.
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3. Hamiltonian Cycle $\leq_p$ TSP:
   Given a graph $G = (V, E)$:

Travelling Salesperson

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   - Set the tour bound to be n.

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   - For each edge $(u, v) \in E$, define $d(u, v) = 1$.
   - For each pair $(u, v) \notin E$, define $d(u, v) = 2$.
   - Set the tour bound to be n.
   - $\Rightarrow$ With a Hamiltonian Cycle in G, the shortest tour will be length n.

Travelling Salesperson

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1. In NP: Certificate that is a tour of the cities.
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   Given a graph $G = (V, E)$:
   - For each v, make a city.
   - For each edge $(u, v) \in E$, define $d(u, v) = 1$.
   - For each pair $(u, v) \notin E$, define $d(u, v) = 2$.
   - Set the tour bound to be n.
   - $\Leftarrow$ If the shortest tour is length n, then no $d(u, v) = 2$ is used, so only edges from the graph are used implying a Hamiltonian cycle in G.

□

Exercise: Show that Hamiltonian Path is NP-complete

Hamiltonian Path

- A simple path in a digraph G that contains all nodes.
- Another sequencing problem.

Exercise: Show that Hamiltonian Path is NP-complete

Theorem 10

Hamiltonian Path is NP*-complete*

Proof.

1. In NP: Certificate is a path in G which can be verified in polynomial time.
2. NP-complete problem: Hamiltonian Cycle.

EXERCISE: SHOW THAT HAMILTONIAN PATH IS NP-COMPLETE

Theorem 10

Hamiltonian Path is NP*-complete*

Proof.

3. Hamiltonian Cycle $\leq_p$ Hamiltonian Path:
   For $G = (V, E)$ create G':

Exercise: Show that Hamiltonian Path is NP-complete

Theorem 10

Hamiltonian Path is NP*-complete*

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3. Hamiltonian Cycle $\leq_p$ Hamiltonian Path:
   For $G = (V, E)$ create G':
   - Choose an arbitrary $v \in V$: $V' = V \setminus \{v\} \cup \{v', v''\}$.

Exercise: Show that Hamiltonian Path is NP-complete

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3. Hamiltonian Cycle $\leq_p$ Hamiltonian Path:
   For $G = (V, E)$ create G':
   - Choose an arbitrary $v \in V$: $V' = V \setminus \{v\} \cup \{v', v''\}$.
   - Initialize $E' = E$:
     - For each edge $(v, w) \in E$: $E' \setminus \{(v, w)\} \cup \{(v', w)\}$.
     - For each edge $(u, v) \in E$: $E' \setminus \{(u, v)\} \cup \{(u, v'')\}$.

EXERCISE: SHOW THAT HAMILTONIAN PATH IS NP-COMPLETE

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     - For each edge $(v, w) \in E$: $E' \setminus \{(v, w)\} \cup \{(v', w)\}$.
     - For each edge $(u, v) \in E$: $E' \setminus \{(u, v)\} \cup \{(u, v'')\}$.
   - A path $v' \to v''$ means Hamiltonian Cycle.

$\square$

Partitioning Problems

3-D Matching

- Given 3 disjoint sets: X, Y, Z (each of size n).
- A set of $m \geq n$ trebles $T \subseteq X \times Y \times Z$.
- Does there exist a set of n trebles from T so that each item is in exactly one of these trebles?

3-D Matching is NP-Complete

Theorem 11

3-D Matching is NP*-Complete.*

Proof.

3-D Matching is NP-Complete

Theorem 11

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Proof.

1. In NP: Certificate is a set of trebles which can be verified in polynomial time.
2. 🤔Which NP-complete problem?

3-D MATCHING IS NP-COMPLETE

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Proof.

1. In NP: Certificate is a set of trebles which can be verified in polynomial time.
2. Use 3SAT.

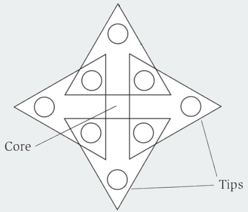
3-D Matching is NP-Complete

Theorem 11

3-D Matching is NP*-Complete.*

Proof.

3. $3SAT \leq_p$ 3-D Matching:
   Consider an arbitrary 3SAT:
   - Variable x_i gadget:
     - Core:
       $A_i = \{a^i_1, \ldots, a^i_{2k}\}$.
     - Tips:
       $B_i = \{b^i_1, \ldots, b^i_{2k}\}$.
     - $t^i_j = (a^i_j, a^i_{j+1}, b^i_j)$ for $j = 1, 2, \ldots, 2k$ (add mod $2k$).



3-D Matching is NP-Complete

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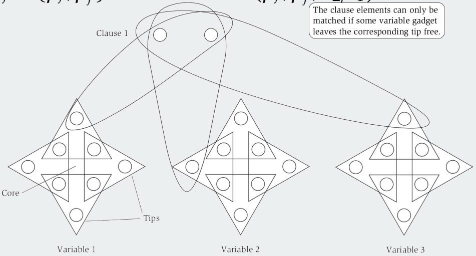
Proof.

3. 3SAT $\leq_p$ 3-D Matching:
   Consider an arbitrary 3SAT:
   - Clause C_j gadget:
     - Add $P_j = \{p_j, p'_j\}$ with trebles: (p_j, p'_j, b^i_{2i-1}) if x_i and (p_j, p'_j, b^i_{2j}) if $\overline{x_i}$.



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3. 3SAT $\leq_p$ 3-D Matching:
   Consider an arbitrary 3SAT:
   - Counting cores: covered by even/odd choice.

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Proof.

3. 3SAT $\leq_p$ 3-D Matching:
   Consider an arbitrary 3SAT:
   - Counting cores: covered by even/odd choice.
   - Counting tips: $n2k$
     - Even/odd tips cover nk.
     - Clauses cover k.
     - $(n-1)k$ uncovered.

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   - $(n-1)k$ clean-up gadgets:
     - $Q_i = \{q_i, q_i'\}$ with treble (q_i, q_i', b) for every tip b.

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     - $(n-1)k$ uncovered.
   - $(n-1)k$ clean-up gadgets:
     - $Q_i = \{q_i, q_i'\}$ with treble (q_i, q_i', b) for every tip b.
   - What are the 3 sets?
     $X = \{a^i_{j\text{ even}}\} \cup \{p_j\} \cup \{q_i\}, Y = \{a^i_{j\text{ odd}}\} \cup \{p_j'\} \cup \{q_i'\}, Z = \{b^i_j\}$.

3-D Matching is NP-Complete

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Proof.

3. 3SAT $\leq_p$ 3-D Matching:
   Consider an arbitrary 3SAT:
   - $\Rightarrow$ For a yes 3SAT, there is a matching that takes the even/odd tip trebles, leaving at least one tip as part of each clause gadget treble. The remaining unmatched tips are match to a clean-up gadget.

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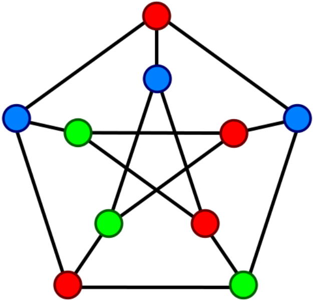
   - $\Rightarrow$ For a yes 3SAT, there is a matching that takes the even/odd tip trebles, leaving at least one tip as part of each clause gadget treble. The remaining unmatched tips are match to a clean-up gadget.
   - $\Leftarrow$ A yes for 3-D Matching from the reduction means that each clause gadget is part of a selected treble, each variable gadget has selected the odd or even tips, and the remaining tips are matched to a clean-up gadget. Each clause will be satisfied by the tip matched by the clause gadget. The even/odd selection for each variable guarantees all variables are assigned 1 or 0.

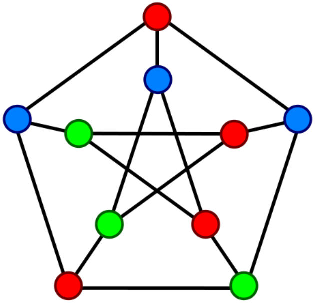
Graph Colouring



Problem

Given a graph G and a bound k, does G have a k-colouring?

Graph Colouring



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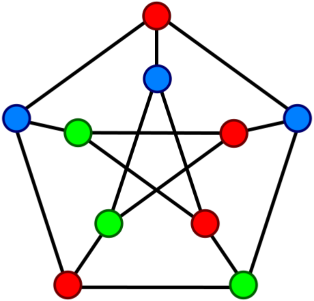
Given a graph G and a bound k, does G have a k-colouring?

***k*-Colour**

- Colouring of the nodes of a graph such that no adjacent nodes have the same colour, using at most k colours.

Graph Colouring



Problem

Given a graph G and a bound k, does G have a k-colouring?

k-Colour

- Colouring of the nodes of a graph such that no adjacent nodes have the same colour, using at most k colours.
- Labelling (partitioning) function $f : V \to \{1, \ldots, k\}$ such that, for every $(u, v) \in E, f(u) \neq f(v)$.

3-COLOURING IS NP-COMPLETE

Theorem 12

3-Colouring is NP*-Complete.*

Proof.

3-Colouring is NP-Complete

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Proof.

1. In NP: Certificate is a colouring of the nodes which can be verified in polynomial time.
2. 🤔Which NP-complete problem?

3-Colouring is NP-Complete

Theorem 12

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1. In NP: Certificate is a colouring of the nodes which can be verified in polynomial time.
2. NP-complete problem: 3SAT.

3-Colouring is NP-Complete

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3. 3SAT $\leq_p$ 3 Colouring:

3-COLOURING IS NP-COMPLETE

Theorem 12

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Proof.

3. 3SAT $\leq_p$ 3 Colouring:
   - For each literal: Nodes v_i and $\overline{v_i}$.

3-COLOURING IS NP-COMPLETE

Theorem 12

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Proof.

3. 3SAT $\leq_p$ 3 Colouring:
   - For each literal: Nodes v_i and $\overline{v_i}$.
   - Nodes T (true), F (false), and B (base).

3-COLOURING IS NP-COMPLETE

Theorem 12

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Proof.

3. 3SAT $\leq_p$ 3 Colouring:

- For each literal: Nodes v_i and $\overline{v_i}$.
- Nodes T (true), F (false), and B (base).
- Edges: $(v_i, \overline{v_i})$,(v_i, B),$(\overline{v_i}, B)$.

3-COLOURING IS NP-COMPLETE

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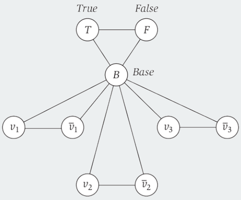

- For each literal: Nodes v_i and $\overline{v_i}$.
- Nodes T (true), F (false), and B (base).
- Edges: $(v_i, \overline{v_i})$,(v_i, B),$(\overline{v_i}, B)$.
- Edges: (T, F),(F, B),(T, B).

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3. 3SAT $\leq_p$ 3 Colouring:
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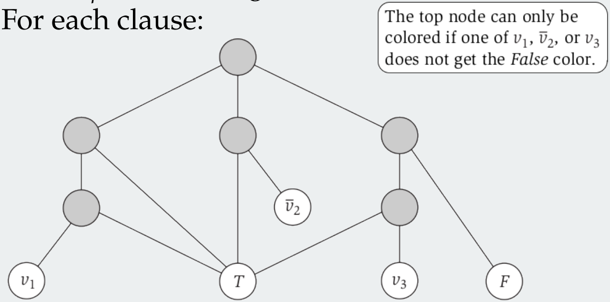

Numerical Problems

Subset Sum Problem

Given a set of n natural numbers $\{w_1, \ldots, w_n\}$ and a target W, is there a subset of the numbers that add up to W?

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Subset Sum Problem

Given a set of n natural numbers $\{w_1, \ldots, w_n\}$ and a target W, is there a subset of the numbers that add up to W?

Dynamic Programming Approach

- We saw an $O(nW)$ algorithm.
- Pseudo-polynomial: W is unbounded, e.g., 2^n.

Subset Sum is NP-Complete

Theorem 13

Subset Sum is NP*-Complete.*

Proof.

Subset Sum is NP-Complete

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Proof.

1. In NP: Certificate is a subset of the numbers which can be verified in polynomial time.
2. 🤔Which NP-complete problem?

Subset Sum is NP-Complete

Theorem 13

Subset Sum is NP*-Complete.*

Proof.

1. In NP: Certificate is a subset of the numbers which can be verified in polynomial time.
2. NP-complete problem: 3-D Matching.

Subset Sum is NP-Complete

Theorem 13

Subset Sum is NP-*Complete.*

Proof.

3. 3-D Matching $\leq_p$ Subset Sum: Exercise: Try it, but tough.

Subset Sum is NP-Complete

Theorem 13

Subset Sum is NP*-Complete.*

Proof.

3. 3-D Matching $\leq_p$ Subset Sum:
   - 3-D Matching: Subsets can be viewed as length $3n$ bit vectors with a 1 indicating that item is in the set.

Subset Sum is NP-Complete

Theorem 13

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Proof.

3. 3-D Matching $\leq_p$ Subset Sum:
   - 3-D Matching: Subsets can be viewed as length $3n$ bit vectors with a 1 indicating that item is in the set.
   - For each treble (i, j, k) from $X \times Y \times Z$ construct a w_t:

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   - For each treble (i, j, k) from $X \times Y \times Z$ construct a w_t:
     - A digits with 1 at i, $n + j$, and $2n + k$.

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   - 3-D Matching: Subsets can be viewed as length $3n$ bit vectors with a 1 indicating that item is in the set.
   - For each treble (i, j, k) from $X \times Y \times Z$ construct a w_t:
     - A digits with 1 at i, $n + j$, and $2n + k$.
     - For base d, $w_t = d^{i-1} + d^{n+j-1} + d^{2n+k-1}$.

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     - Set base $d = m + 1$ to avoid addition carry overs.

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     - For base d, $w_t = d^{i-1} + d^{n+j-1} + d^{2n+k-1}$.
     - Set base $d = m + 1$ to avoid addition carry overs.
   - Set $W = \sum_0^{3n-1}(m+1)^i$ which corresponds to have each item exactly once.

□

Constraint Satisfaction Problems

Not All Equal 4SAT (NAE 4SAT)

Given a 4SAT formula, is there an assignment to the literals such that every clause contains at least one true term and at least one false term.

NAE 4SAT is NP-Complete

Theorem 14

NAE 4SAT *is* NP-*Complete.*

Proof.

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Proof.

1. In NP:
2.

NAE 4SAT is NP-Complete

Theorem 14

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Proof.

1. In NP: Certificate is an assignment of values to the literals which can be verified in polynomial time.
2.

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Proof.

1. In NP: Certificate is an assignment of values to the literals which can be verified in polynomial time.
2. NP-complete problem: 3SAT.

NAE 4SAT is NP-Complete

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Proof.

3. 3SAT $\leq_p$ NAE 4SAT:

NAE 4SAT IS NP-COMPLETE

Theorem 14

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Proof.

3. 3SAT $\leq_p$ NAE 4SAT:
   - Add a literal v to every 3SAT clause of Φ to create a NEA – 4SAT formula Φ'.

NAE 4SAT is NP-Complete

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3. 3SAT $\leq_p$ NAE 4SAT:
   - Add a literal v to every 3SAT clause of Φ to create a NEA – 4SAT formula Φ'.

   Reduction correctness:

NAE 4SAT IS NP-COMPLETE

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Proof.

3. 3SAT $\leq_p$ NAE 4SAT:
   - Add a literal v to every 3SAT clause of Φ to create a NEA – 4SAT formula Φ'.

   Reduction correctness:
   - $\Rightarrow$: If 3SAT Φ is true, $\exists$ an assignment where every clause in Φ has ≥ 1 true value. Set $v = 0$ and Φ' is satisfied.

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   Reduction correctness:
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   - $\Leftarrow$: If NAE 4SAT is true:

NAE 4SAT IS NP-COMPLETE

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   Reduction correctness:
   - $\Rightarrow$: If 3SAT Φ is true, $\exists$ an assignment where every clause in Φ has ≥ 1 true value. Set $v = 0$ and Φ' is satisfied.
   - $\Leftarrow$: If NAE 4SAT is true:
     - Case 1: $v = 0$. Each clause in Φ' has at least 1 term that is not v set to true $\implies$ Φ is satisfied.

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   - Add a literal v to every 3SAT clause of Φ to create a NEA – 4SAT formula Φ'.

   Reduction correctness:
   - $\Rightarrow$: If 3SAT Φ is true, $\exists$ an assignment where every clause in Φ has ≥ 1 true value. Set $v = 0$ and Φ' is satisfied.
   - $\Leftarrow$: If NAE 4SAT is true:
     - Case 1: $v = 0$. Each clause in Φ' has at least 1 term that is not v set to true $\implies$ Φ is satisfied.
     - Case 2: $v = 1$. Each clause in Φ' has at least 1 term that is not v set to false $\implies$ Φ is satisfied by the complement of the assignment that satisfies Φ'.

$\square$

Taxonomy of Hard Problems

Packing Problems

- **Independent Set**
- Set Packing
- **Clique** (in discussion)

Covering Problems

- **Vertex Cover**
- Set Cover

Sequencing Problems

- TSP
- Hamiltonian Cycle
- Hamiltonian Path

Taxonomy of Hard Problems

Partitioning Problems

- 3-D Matching
- Graph Colouring

Numerical Problems

- **Subset Sum**
- Knapsack

Constraint Satisfaction Problems

- **3SAT**
- *CSAT*
- *NAE 4SAT*

coNP

Asymmetry of NP

Efficient Certifier Asymmetry

Given an instance s of problem X:

- For any t, $B(s,t)$ = yes implies yes-instance.
- For all t, $B(s,t)$ = no implies no-instance.

ASYMMETRY OF NP

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Complimentary Problem

For every problem X, there is a complementary problem $\overline{X}$:

- For all input s, $s \in X$ iff $s \notin \overline{X}$.

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Complimentary Problem

For every problem X, there is a complementary problem $\overline{X}$:

- For all input s, $s \in X$ iff $s \notin \overline{X}$.
- Note that, if $X \in \mathrm{P}$, then $\overline{X} \in \mathrm{P}$.

coNP

Complexity Class coNP

A problem $X \in \text{coNP}$ iff $\overline{X} \in \text{NP}$.

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Proof.

Contra-positive: Prove it!

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- $X \in \text{coNP} \rightarrow \overline{X} \in \text{NP} \rightarrow \overline{X} \in \text{P} \rightarrow X \in \text{P} \rightarrow X \in \text{NP}$.

$\square$

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PSPACE

Beyond Time

Complexity Class PSPACE

Set of all problems that can be solved using polynomial space.

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Theorem 16

$\mathrm{P} \subseteq \mathrm{PSPACE}$

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Theorem 16

P $\subseteq$ PSPACE

Theorem 17

NP $\subseteq$ PSPACE

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Theorem 16

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Proof.

- For 3SAT, a bit vector can encode an assignment.
- We can try all bit vectors with one n-length vector in memory:
  - Start with 0 until $2^n - 1$, adding 1 at each iteration.

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Theorem 16

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Proof.

- For 3SAT, a bit vector can encode an assignment.
- We can try all bit vectors with one n-length vector in memory:
  - Start with 0 until $2^n - 1$, adding 1 at each iteration.
- Since 3SAT $\in$ PSPACE and is NP-complete, for any $Y \in$ NP, $Y \leq_p$ 3SAT and solve in PSPACE. □

PROTOTYPICAL PSPACE PROBLEM

Let $\Phi(x_1, \ldots, x_n)$ be a conjunction of k disjunction of n variables (like SAT).

Prototypical PSPACE Problem

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Quantified SAT

- $\exists x_1 \forall x_2 \exists x_3 \cdots Q_n x_n \Phi(x_1, \ldots, x_n)$ (Prenex normal form).
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Theorem 18

QSAT *is* PSPACE-*complete.*

Appendix

References

Image Sources I

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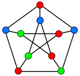
https://en.wikipedia.org/wiki/Stephen_Cook



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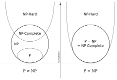

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