CS 577 - Graph Algorithms Part (A)

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WISCONSIN
UNIVERSITY OF WISCONSIN–MADISON

Shortest Path

Finding the Shortest Path

Problem Definition

We have a directed graph $G = (V, E)$, where $|V| = n$ and $|E| = m$ and a node s that has a path to every other node in V. For each edge e, $\ell_e \geq 0$ is the length of the edge.

- What is the shortest path from s to each other node?

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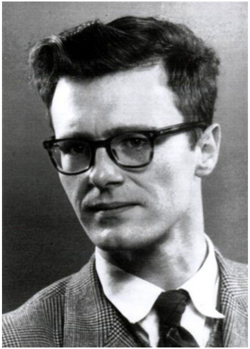

Edsger Dijkstra, 1956
Dijkstra's shortest path fame

Dijkstra's

Algorithm: *Dijkstra's*

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Initialize $S = \{s\}$ and $d(s) = 0$
while $S \neq V$ **do**
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$$d'(v) = \min_{e=(u,v):u \in S} \{d(u) + \ell_e\}$$

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How is it greedy?

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How is it greedy?
Which technique to prove optimality?

Correctness of Dijkstra's

Theorem 1

Consider the S at any point in the execution of Dijkstra's. For each $u \in S$, the path P_u is a shortest $s - u$ path.

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- For $|S| = 1$, the claim follows trivially as $S = \{s\}$.
- By the induction hypothesis, for $|S| = k$, P_u is the shortest $s - u$ path for all $u \in S$.

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- In step $k + 1$, we add v.
  - By definition, P_v is shortest path connected to S by one edge.

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- In step $k + 1$, we add v.
  - By definition, P_v is shortest path connected to S by one edge.
  - Since P_u is a shortest path to u, P_v is the shortest path to v when considering only the nodes of S.
  - Moreover, there cannot be a shorter path to v passing through another node $y \notin S$ else y that would be added at $k + 1$.

□

DIJKSTRA'S OBSERVATIONS

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- Negative edge weights, where does it fail?

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  - Weighted (continuous) BFS

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- How can we get $O(n + n \log n)$?

Shortest Path-Negative Side of the Moon

Shortest Path

Going Negative

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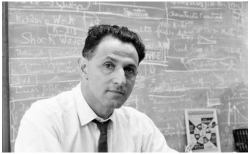
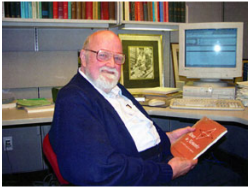


Richard Bellman



L R Ford Jr.

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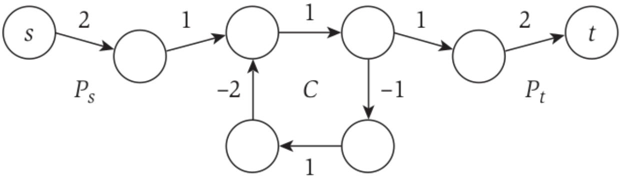
Why no negative cycles?



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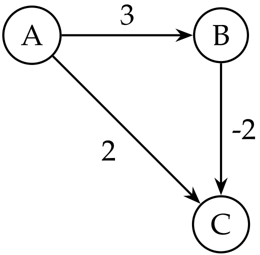
- Lose guarantee that minimum edge between S and $V \setminus S$ is part of minimum path.



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- A path with x edges: Cost increases $x \cdot \beta$.
- Solution in new graph is not guaranteed to be optimal in original graph.

Bellman-Ford

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🤔Build the Bellman equation.

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$$M[i][v] = \min\{M[i-1][v], \min_{w \in V}\{M[i-1][w] + c_{vw}\}\},$$

where $c_{vw} = \infty$ if no edge from v to w.

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Worst Case: n nodes

- 🤔 # of Cells:

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- For each node v, we only need to consider outgoing edges to w (denoted by η_v).
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Space Saving: $O(n)$.

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- Recovery of actual path: An additional array $\mathit{first}[v]$ that maintains the first hop from v to t.

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Augmented Graph for Negative Cycle Finding

- Add a node t with an incoming edge from all other nodes with cost 0.
- Run Bellman-Ford from any node s to t until number of edges n.
- If, for some v, $M[n][v] \neq M[n-1][v]$, then there is a negative cycle.

MST

Minimum Spanning Tree Problem

MST Problem

Let $G = (V, E)$ be a connected graph, where $|V| = n$ and $|E| = m$. For each edge e, $c_e > 0$ is the cost of the edge.

- Find an edge set $F \subseteq E$ with minimum cost that keeps the graph connected. That is, F should minimize $\sum_{e \in F} c_e$.

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Proof.

- By the definition of the problem, T must be connected.
- By way of contradiction, assume that T has a cycle C. Remove any edge from C resulting in a graph T'. T' is still connect and has a cost less than T.

□

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Kruskal's (1956) Algorithm

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- Insert edges unless insertion would create a cycle.

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Jarník's (1929), Kruskal's (1956), Prim's (1957), Loberman and Weinberger (1957), Dijkstra's (1958) Algorithm

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🤔What greedy heuristic might work?

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WLOG (without loss of generality)

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ANALYZING MST HEURISTICS

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- That is, Prim's algorithm does exactly what Lemma 3 describes.

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Reverse-Delete is Optimal

Reverse-Delete (Kruskal's 1956) Algorithm

- Sort edges by cost from highest to lowest.
- Remove edges unless graph would become disconnected.

How should we prove that it produces an MST?

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Let C be any cycle in G, and let e be the most expensive edge of C. Then, e is not in any MST of G.

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Priority Queue (min-heap)

- `ExtractMin` $(O(1))$: $n-1$ times.
- `ChangeKey` $(O(\log(n)))$: m times.

Overall: $O(m \log(n))$

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Union-Find Data Structure

- `Find(x)`: Finds the set containing x. ($O(\log n)$ can be $O(\alpha(n))$)
- `Union(x,y)`: Joins two sets x and y. ($O(1)$)

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`node`	`rank`	`parent`

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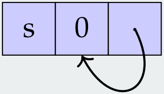
Initializing Data Structure for Kruskal's

For each node s, create a singleton set. That is each container has rank 0 and points to itself.



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Find(x): $O(\log n)$

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Union(x,y): $O(1)$

- (WLOG) x.rank $\geq y$.rank:
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- If x.rank $= y$.rank:
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- By using rank, we maintain balanced sets if we start with balanced sets.

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TH: How many Find and Unions?

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Union-Find Data Structure

- `Find(x)`: $2m$ times $O(\log n)$ (can be $O(\alpha(n))$).
- `Union(x,y)`: $n - 1$ times $O(1)$.

Graph Exploration Overview

BFS and DFS

- Traverses a graph G starting from some node s.
- Builds a tree T.
- No guarantee on any distance measure.

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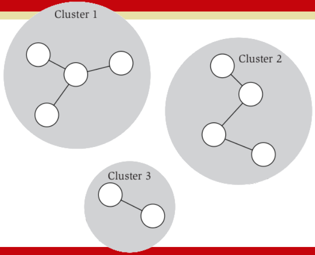
MST Algorithms

- Explores a graph G edges.
- Builds a tree T.
- T is minimum cost to connect all nodes in G.

Clustering

k-Clustering



Maximizing Spacing Problem

- A universe $\mathcal{U} := \{p_1, \ldots, p_n\}$ of n objects.
- Distance function $d : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ such that, for all $p_i, p_j \in \mathcal{U}$:
  - $d(p_i, p_i) = 0$
  - $d(p_i, p_j) > 0$
  - $d(p_i, p_j) = d(p_j, p_i)$
- Objective: Partition $\mathcal{U}$ into k non-empty groups $\mathcal{C} := C_1, \ldots, C_k$ with maximum spacing:

$$\text{maximize} \min_{C_i, C_j \in \mathcal{C}} \min_{u \in C_i, v \in C_j} d(u, v)$$

Algorithm Design

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Algorithm Design

Algorithm

- Build an MST.
- Remove $k - 1$ largest edges.

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k-Clusters at max spacing?

- Start with a tree, remove $k-1$ edges: We get a forest of k trees.
- By definition largest edges are removed so max spacing.

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Which MST algorithm?

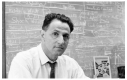
Kruskal's ($O(m \log n)$ which is $O(n^2 \log n)$ for clustering):

- Merge sets from lowest to most expensive edges.
- Stop when we have k sets.

Appendix

References

Image Sources I



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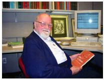

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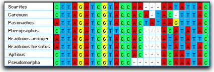

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