CS540
Uninformed Search

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Main messages

• Many AI problems can be formulated as search.

• Iterative deepening is good when you don’t know much.
PROBLEM: The boat only holds two, but you can't leave the goat with the cabbage or the wolf with the goat.

SOLUTION: 1. Take the goat across.
2. Return alone.
3. Take the cabbage across.
4. Leave the wolf.

Why did you have a wolf?
The search problem

- **State space** $S$ : all valid configurations
- **Initial states (nodes)** $I=\{(\text{CSDF,})\} \subseteq S$
  - Where’s the boat?
- **Goal states** $G=\{(,\text{CSDF})\} \subseteq S$
- **Successor function** $\text{succs}(s) \subseteq S$ : states reachable in one step (one arc) from $s$
  - $\text{succs}((\text{CSDF,})) = \{(\text{CD, SF})\}$
  - $\text{succs}((\text{CDF,S})) = \{(\text{CD,FS}, \text{D,CFS}, \text{C, DFS})\}$
- **Cost** $(s,s’) = 1$ for all arcs. (weighted later)
- The search problem: find a solution path from a state in $I$ to a state in $G$.
  - Optionally minimize the cost of the solution.
Search examples

- 8-puzzle

States = configurations
successor function = up to 4 kinds of movement
Cost = 1 for each move
Search examples

• Water jugs: how to get 1?

• Goal? (How many goal states?)
• Successor function: fill up (from tap or other jug), empty (to ground or other jug)
Search examples

- Route finding (state? Successors? Cost weighted)
8-queens

- State: complete configuration vs. column-by-column

- Tree instead of graph
A directed graph in state space

In general there will be many generated, but unexpanded states at any given time.
One has to choose which one to expand next.
Different search strategies

- The generated, but not yet expanded states form the fringe (OPEN).
- The essential difference is which one to expand first.
- Deep or shallow?
Uninformed search on trees

• **Uninformed** means we only know:
  – The goal test
  – The `succs()` function

• But **not** which non-goal states are better: that would be informed search (next lecture).

• For now, we also assume `succs()` graph is a tree.
  ▪ Won’t encounter repeated states.
  ▪ We will relax it later.

• Search strategies: BFS, UCS, DFS, IDS, BIBFS

• Differ by what un-expanded nodes to expand
Breadth-first search (BFS)

Expand the shallowest node first
- Examine states \textbf{one} step away from the initial states
- Examine states \textbf{two} steps away from the initial states
- and so on…
Breadth-first search (BFS)

Use a queue (First-in First-out)

1. en_queue(Initial states)
2. While (queue not empty)
3. s = de_queue()
4. if (s==goal) success!
5. T = succs(s)
6. en_queue(T)
7. endwhile
Breadth-first search (BFS)

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queue (fringe, OPEN)
\[ \rightarrow [A] \rightarrow \]
Breadth-first search (BFS)

Use a queue (First-in First-out)
1. `en_queue(Initial states)`
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queue (fringe, OPEN)  \[ \rightarrow [CB] \rightarrow A \]
Breadth-first search (BFS)

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queue (fringe, OPEN) → [EDC] → B
Breadth-first search (BFS)

Use a queue (First-in First-out)
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2. While (queue not empty)
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4. if (s==goal) success!
5. `T = succs(s)`
6. `en_queue(T)`
7. `endWhile`

Queue (fringe, OPEN)
→[GFED] → C

If G is a goal, we've seen it, but we don't stop!
Breadth-first search (BFS)

Use a **queue** (First-in First-out)

- `en_queue(Initial states)`
- **While** (queue not empty)
  - `s = de_queue()`
  - if (`s==goal`) success!
  - `T = succs(s)`
  - for `t in T`: `t.prev=s`
  - `en_queue(T)`
- **endWhile**

Looking stupid? Indeed. But let’s be consistent…

... until much later we pop G.

We need **back pointers** to recover the solution path.
Performance of BFS

• Assume:
  ▪ the graph may be infinite.
  ▪ Goal(s) exists and is only finite steps away.
• Will BFS find at least one goal?
• Will BFS find the least cost goal?
• Time complexity?
  ▪ # states generated
  ▪ Goal $d$ edges away
  ▪ Branching factor $b$
• Space complexity?
  ▪ # states stored
Performance of BFS

Four measures of search algorithms:

- **Completeness** (not finding all goals): yes, BFS will find a goal.

- **Optimality**: yes if edges cost 1 (more generally positive non-decreasing in depth), no otherwise.

- **Time complexity** (worst case): goal is the last node at radius $d$.
  - Have to generate all nodes at radius $d$.
  - $b + b^2 + \ldots + b^d \sim O(b^d)$

- **Space complexity** (bad)
  - Back pointers for all generated nodes $O(b^d)$
  - The queue / fringe (smaller, but still $O(b^d)$)
What’s in the fringe (queue) for BFS?

• Convince yourself this is $O(b^d)$
### Performance of search algorithms on trees

- **b**: branching factor (assume finite)
- **d**: goal depth

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1. Edge cost constant, or positive non-decreasing in depth
Performance of BFS

Four measures of search algorithms:

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  - Have to generate all nodes at radius $d$.
  - $b + b^2 + \ldots + b^d \sim O(b^d)$
- **Space complexity** (bad, Figure 3.11)
  - Back points for all generated nodes $O(b^d)$
  - The queue (smaller, but still $O(b^d)$)

**Solution:** Uniform-cost search
Uniform-cost search

- Find the least-cost goal
- Each node has a path cost from start (= sum of edge costs along the path). Expand the least cost node first.
- Use a priority queue instead of a normal queue
  - Always take out the least cost item
  - Remember heap? time $O(\log(\#\text{items in heap}))$

That’s it*

* Complications on graphs (instead of trees). Later.
Uniform-cost search (UCS)

- Complete and optimal (if edge costs $\geq \varepsilon > 0$)
- Time and space: can be much worse than BFS
  - Let $C^*$ be the cost of the least-cost goal
  - $O(b^{C^*/\varepsilon})$, possibly $C^*/\varepsilon >> d$
## Performance of search algorithms on trees

*b: branching factor (assume finite)  d: goal depth*

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1. edge cost constant, or positive non-decreasing in depth
   - edge costs $\geq \varepsilon > 0$. $C^*$ is the best goal path cost.
General State-Space Search Algorithm

function general-search(problem, QUEUEING-FUNCTION)

;; problem describes the start state, operators, goal test, and
;; operator costs
;; queueing-function is a comparator function that ranks two states
;; general-search returns either a goal node or "failure"

nodes = MAKE-QUEUE(MAKE-NODE(problem.INITIAL-STATE))
loop
    if EMPTY(nodes) then return "failure"
    node = REMOVE-FRONT(nodes)
    if problem.GOAL-TEST(node.STATE) succeeds then return node
    nodes = QUEUEING-FUNCTION(nodes, EXPAND(node, problem.OPERATORS))
    ;; succ(s)=EXPAND(s, OPERATORS)
    ;; Note: The goal test is NOT done when nodes are generated
    ;; Note: This algorithm does not detect loops
end
Recall the bad space complexity of BFS

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- **Space complexity** (bad, Figure 3.11)
  - Back points for all generated nodes $O(b^d)$
  - The queue (smaller, but still $O(b^d)$)

Solution: Uniform-cost search

Solution: Depth-first search
Depth-first search

Expand the deepest node first

1. Select a direction, go deep to the end
2. Slightly change the end
3. Slightly change the end some more…

fan
Depth-first search (DFS)

Use a stack (First-in Last-out)

1. push(Initial states)
2. While (stack not empty)
3. s = pop()
4. if (s==goal) success!
5. T = succs(s)
6. push(T)
7. endwhile
What’s in the fringe for DFS?

- $m = \text{maximum depth of graph from start}$
- $m(b-1) \sim O(mb)$ (Space complexity)

- “backtracking search” even less space
  - generate siblings (if applicable)

\[c.f. \text{ BFS } O(b^d)\]
What’s wrong with DFS?

- Infinite tree: may not find goal (incomplete)
- May not be optimal
- Finite tree: may visit almost all nodes, time complexity $O(b^m)$

c.f. BFS $O(b^d)$
### Performance of search algorithms on trees

- **b**: branching factor (assume finite)
- **d**: goal depth
- **m**: graph depth

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1. edge cost constant, or positive non-decreasing in depth
   - edge costs \(\geq \varepsilon > 0\). \(C^*\) is the best goal path cost.
How about this?

1. DFS, but stop if path length > 1.
2. If goal not found, repeat DFS, stop if path length > 2.
3. And so on…

fan within ripple
Iterative deepening

• Search proceeds like BFS, but fringe is like DFS
  ▪ Complete, optimal like BFS
  ▪ Small space complexity like DFS
• A huge waste?
  ▪ Each deepening repeats DFS from the beginning
  ▪ No! \( db + (d-1)b^2 + (d-2)b^3 + \ldots + b^d \sim O(b^d) \)
  ▪ Time complexity like BFS
• Preferred uninformed search method
## Performance of search algorithms on trees

- **b**: branching factor (assume finite)
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1. **edge cost constant, or positive non-decreasing in depth**
   - edge costs $\geq \varepsilon > 0$. $C^*$ is the best goal path cost.
Bidirectional search

• Breadth-first search from both start and goal
• Fringes meet
• Generates $O(b^{d/2})$ instead of $O(b^d)$ nodes
Bidirectional search

• But
  ▪ The fringes are $O(b^{d/2})$
  ▪ How do you start from the 8-queens goals?
## Performance of search algorithms on trees

**b**: branching factor (assume finite)  
**d**: goal depth  
**m**: graph depth

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1. edge cost constant, or positive non-decreasing in depth  
   - edge costs \(\geq \varepsilon > 0\). \(C^*\) is the best goal path cost.  
   - both directions BFS; not always feasible.
If state space graph is not a tree

• The problem: repeated states

• Ignore the danger of repeated states: wasteful (BFS) or impossible (DFS). Can you see why?

• How to prevent it?
If state space graph is not a tree

• We have to remember already-expanded states (CLOSED).

• When we take out a state from the fringe (OPEN), check whether it is in CLOSED (already expanded).
  ▪ If yes, throw it away.
  ▪ If no, expand it (add successors to OPEN), and move it to CLOSED.
If state space graph is not a tree

• **BFS:**
  - Still $O(b^d)$ space complexity, not worse

• **DFS:**
  - Known as Memorizing DFS (**MEMDFS**)  
    - Space and time now $O(\min(N, b^M))$ – much worse!  
    - $N$: number of states in problem  
    - $M$: length of longest cycle-free path from start to anywhere

  - Alternative: Path Check DFS (**PCDFS**): remember only expanded states on current path (from start to the current node)
    - Space $O(M)$  
    - Time $O(b^M)$
Path Checking DFS

1. Maintain a “prefix” path from root to current node, initially empty.
2. Pop a state $s$. If $s$ in prefix, skip to next pop
3. Goal-checking $s$.
4. $s$ comes with a backpointer to its parent $p$. The prefix should contain $p$ somewhere as in initial, ..., $p$, ...
5. Remove everything after $p$ and put $s$ there, so prefix is now initial, ..., $p$, $s$.
6. When you generate a successor $t$ of $s$, check if $t$ is in prefix or stack. If no, push $t$ to the stack; if yes, do not push it.
Example

(All edges are directed, pointing downwards)
Nodes expanded by:

- Depth-First Search: S A D E G
  Solution found: S A G

- Breadth-First Search: S A B C D E G
  Solution found: S A G

- Uniform-Cost Search: S A D B C E G
  Solution found: S B G (This is the only uninformed search that worries about costs.)

- Iterative-Deepening Search: S A B C S A D E G
  Solution found: S A G
# Depth-First Search

<table>
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<tr>
<th>expanded node</th>
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<td>S</td>
<td>{ A, B, C }</td>
</tr>
<tr>
<td>A</td>
<td>{ D, E, G, B, C }</td>
</tr>
<tr>
<td>D</td>
<td>{ E, G, B, C }</td>
</tr>
<tr>
<td>E</td>
<td>{ G, B, C }</td>
</tr>
<tr>
<td>G</td>
<td>{ B, C }</td>
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Solution path found is S A G  <-- this G has cost 10
Number of nodes expanded (including goal node) = 5
### Breadth-First Search

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<td>{ A B C }</td>
</tr>
<tr>
<td>A</td>
<td>{ B C D E G }</td>
</tr>
<tr>
<td>B</td>
<td>{ C D E G G' }</td>
</tr>
<tr>
<td>C</td>
<td>{ D E G G' G'' }</td>
</tr>
<tr>
<td>D</td>
<td>{ E G G' G'' }</td>
</tr>
<tr>
<td>E</td>
<td>{ G G' G'' }</td>
</tr>
<tr>
<td>G</td>
<td>{ G' G'' }</td>
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</tbody>
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Solution path found is S A G  <-- this G also has cost 10
Number of nodes expanded (including goal node) = 7
Uniform-Cost Search

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<tr>
<td>S</td>
<td>{ S }</td>
</tr>
<tr>
<td>A</td>
<td>{ A(1), B(5), C(8) }</td>
</tr>
<tr>
<td>D</td>
<td>{ D(4), B(5), C(8), E(8), G(10) } (note, we don’t return G)</td>
</tr>
<tr>
<td>B</td>
<td>{ B(5), C(8), E(8), G(10) }</td>
</tr>
<tr>
<td>C</td>
<td>{ C(8), E(8), G(9), G(10) }</td>
</tr>
<tr>
<td>E</td>
<td>{ E(8), G(9), G(10), G(13) }</td>
</tr>
<tr>
<td>G</td>
<td>{ G(9), G(10), G(13) }</td>
</tr>
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</table>

Solution path found is S B G <-- this G has cost 9, not 10
Number of nodes expanded (including goal node) = 7
What you should know

• Problem solving as search: state, successors, goal test
• Uninformed search
  ▪ Breadth-first search
    • Uniform-cost search
  ▪ Depth-first search
  ▪ Iterative deepening
  ▪ Bidirectional search
• Can you unify them (except bidirectional) using the same algorithm, with different priority functions?
• Performance measures
  ▪ Completeness, optimality, time complexity, space complexity