First Order Logic
Part 2

Yingyu Liang

yliang@cs.wisc.edu

Computer Sciences Department

University of Wisconsin, Madison

[Based on slides from Burr Settles and Jerry Zhu]

Thinking in logical sentences

Convert the following sentences into FOL:

- “Elmo is a monster.”
  - What is the constant? Elmo
  - What is the predicate? Is a monster
  - Answer: monster(Elmo)

- “Tinky Winky and Dipsy are teletubbies”

- “Tom, Jerry or Mickey is not a mouse.”

Thinking in logical sentences

We can also do this with relations:

- “America bought Alaska from Russia.”
  - What are the constants?
    - America, Alaska, Russia
  - What are the relations?
    - Bought
  - Answer: bought(America, Alaska, Russia)
- “Warm is between cold and hot.”
- “Jerry and Jing are married.”

Thinking in logical sentences

Now let's think about quantifiers:

- "Jerry likes everything."
  - What's the constant?
    - Jerry
  - Thing?
    - Just use a variable x
  - Everything?
    - Universal quantifier
  - Answer: `∀x likes(Jerry, x)`
  - *i.e.* `likes(Jerry, IceCream) ∧ likes(Jerry, Jing) ∧ likes(Jerry, Armadillos) ∧ …`
- "Jerry likes something."
- "Somebody likes Jerry."

Thinking in logical sentences

We can also have multiple quantifiers:

- “somebody heard something.”
  - What are the variables?
    - Somebody, something
  - How are they quantified?
    - Both are existential
  - Answer: `∃x,y heard(x,y)`

- “Everybody heard everything.”

- “Somebody did not hear everything.”

Thinking in logical sentences

Let's allow more complex quantified relations:

- "All stinky shoes are allowed."
  - How are ideas connected?
    - Being a shoe and being stinky implies it's allowed
  - Answer: $\forall$**x shoe(x) ∧ stinky(x) ⇒ allowed(x)**
- "No stinky shoes are allowed."
  - Answers:
  - $\forall$**x shoe(x) ∧ stinky(x) ⇒ ¬allowed(x)**
  - ¬∃**x shoe(x) ∧ stinky(x) ∧ allowed(x)**
  - ¬∃**x shoe(x) ∧ stinky(x) ⇒ allowed(x) (?)**

Thinking in logical sentences

- "No stinky shoes are allowed."
  - $\neg\exists x$ `shoe(x)` $\wedge$ `stinky(x)` $\Rightarrow$ `allowed(x)` `(?)`
  - $\neg\exists x$ $\neg$`(shoe(x)` $\wedge$ `stinky(x))` $\vee$ `allowed(x)`
  - $\forall x$ $\neg$ `(`$\neg$`(shoe(x)` $\wedge$ `stinky(x))` $\vee$ `allowed(x))`
  - $\forall x$ `(shoe(x)` $\wedge$ `stinky(x))` $\wedge$ $\neg$`allowed(x)`
- But this says "Jerry is a stinky shoe and Jerry is not allowed."
- How about

  $\forall x$ `allowed(x)` $\Rightarrow$ $\neg$ `(shoe(x)` $\wedge$ `stinky(x))`

Thinking in logical sentences

And some more complex relations:

- “No one sees everything.”
- Answer: $\neg\exists x\ \forall y\ \texttt{sees}(x,y)$

- Equivalently: “Everyone doesn’t see something.”
- Answer: $\forall x\ \exists y\ \neg\texttt{sees}(x,y)$

- “Everyone sees nothing.”
- Answer: $\forall x\ \neg\exists y\ \texttt{sees}(x,y)$

Thinking in logical sentences

And some *really* complex relations:

- "Any good amateur can beat some professional."
  - Ingredients: x, amateur(x), good(x), y, professional(y), beat(x,y)
  - Answer:

$$\forall x\ [\{amateur(x) \land good(x)\} \Rightarrow \exists y\ \{professional(y) \land beat(x,y)\}]$$

- "Some professionals can beat all amateurs."
  - Answer:

$$\exists x\ [professional(x) \land \forall y\ \{amateur(y) \Rightarrow beat(x,y)\}]$$

Thinking in logical sentences

We can throw in functions and equalities, too:

- “Jerry and Jing are the same age.”
  - Are functional relations specified?
  - Are equalities specified?
  - Answer: **`age(Jerry) = age(Jing)`**
- “There are exactly two shoes.”
  - ?

Thinking in logical sentences

- “There are exactly two shoes.”
  - First try:

$\exists$**x** $\exists$**y** **shoe(x)** $\wedge$ **shoe(y)**

Thinking in logical sentences

- "There are exactly two shoes."
  - First try:

$\exists x\ \exists y\ shoe(x) \wedge shoe(y)$

  - Second try:

$\exists x\ \exists y\ shoe(x) \wedge shoe(y) \wedge \neg(x=y)$

Thinking in logical sentences

- "There are exactly two shoes."
  - First try:

$\exists x\ \exists y\ shoe(x) \wedge shoe(y)$

  - Second try:

$\exists x\ \exists y\ shoe(x) \wedge shoe(y) \wedge \neg(x=y)$

  - Third try:

$$\exists x\ \exists y\ shoe(x) \wedge shoe(y) \wedge \neg(x=y) \wedge \\ \forall z\ (shoe(z) \Rightarrow (x=z) \vee (y=z))$$

Thinking in logical sentences

- Interesting words: **always, sometimes, never**
  - "Good people *always* have friends."

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    $\forall$**`x person(x)`** $\wedge$ **`good(x)`** $\Rightarrow$ $\exists$**`y(friend(x,y))`**
  - “Busy people *sometimes* have friends.”

Thinking in logical sentences

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  - "Good people *always* have friends."

    $\forall$**x person(x) $\wedge$ good(x) $\Rightarrow$ $\exists$y(friend(x,y))**
  - "Busy people *sometimes* have friends."

    $\exists$**x person(x) $\wedge$ busy(x) $\wedge$ $\exists$y(friend(x,y))**
  - "Bad people *never* have friends."

Thinking in logical sentences

- Interesting words: **always, sometimes, never**
  - "Good people *always* have friends."

    $\forall x$ **person(x) $\wedge$ good(x) $\Rightarrow$ $\exists$y(friend(x,y))**
  - "Busy people *sometimes* have friends."

    $\exists x$ **person(x) $\wedge$ busy(x) $\wedge$ $\exists$y(friend(x,y))**
  - "Bad people *never* have friends."

    $\forall x$ **person(x) $\wedge$ bad(x) $\Rightarrow$ $\neg\exists$y(friend(x,y))**

Thinking in logical sentences

Tricky sentences

- "x is above y if and only if x is directly on the top of y, or else there is a pile of one or more other objects directly on top of one another, starting with x and ending with y."

Thinking in logical sentences

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- "x is above y if and only if x is directly on the top of y, or else there is a pile of one or more other objects directly on top of one another, starting with x and ending with y."

$\forall$**x** $\forall$**y above(x,y)** $\Leftrightarrow$

[onTop(x,y) $\vee$ $\exists$**z{onTop(x,z)** $\wedge$ **above(z,y)}]**

Professor Snape's Puzzle

Danger lies before you, while safety lies behind,
Two of us will help you, whichever you would find,
One among us seven will let you move ahead,
Another will transport the drinker back instead,
Two among our number hold only nettle-wine,
Three of us are killers, waiting hidden in line
Choose, unless you wish to stay here forevermore
To help you in your choice, we give you these clues four:
First, however slyly the poison tries to hide
You will always find some on nettle wine's left side
Second, different are those who stand at either end
But if you would move onward, neither is your friend;
Third as you see clearly, all are different size
Neither dwarf nor giant hold death in their insides;
Fourth, the second left and the second on the right
Are twins once you taste them, though different at first sight.

1. $\exists x\ A(x) \wedge (\forall y\ A(y) \Rightarrow x=y)$
2. $\exists x\ B(x) \wedge (\forall y\ B(y) \Rightarrow x=y)$
3. $\exists x \exists y\ W(x) \wedge W(y) \wedge \neg(x=y) \wedge (\forall z\ W(z) \Rightarrow z=x \vee z=y)$
4. $\forall x\ \neg(A(x) \vee B(x) \vee W(x)) \Rightarrow P(x)$
5. $\forall x \forall y\ W(x) \wedge L(y,x) \Rightarrow P(y)$
6. $\neg(P(b1) \wedge P(b7))$
7. $\neg(W(b1) \wedge W(b7))$
8. $\neg\ A(b1)$
9. $\neg\ A(b7)$
10. $\neg\ P(b3)$
11. $\neg\ P(b6)$
12. $(P(b2) \wedge P(b6)) \vee (W(b2) \wedge W(b6))$