Bayesian Networks Part 2

Yingyu Liang
Computer Sciences 760
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http://pages.cs.wisc.edu/~yliang/cs760/

Some of the slides in these lectures have been adapted/borrowed from materials developed by Mark Craven, David Page, Jude Shavlik, Tom Mitchell, Nina Balcan, Matt Gormley, Elad Hazan, Tom Dietterich, and Pedro Domingos.

Goals for the lecture

you should understand the following concepts

- missing data in machine learning
    - hidden variables
    - missing at random
    - missing systematically
- the EM approach to imputing missing values in Bayes net parameter learning

- the Chow-Liu algorithm for structure search

Missing data

- Commonly in machine learning tasks, some feature values are missing

- some variables may not be observable (i.e. *hidden*) even for training instances

- values for some variables may be *missing at random*: what caused the data to be missing does not depend on the missing data itself
    - e.g. someone accidentally skips a question on an questionnaire
    - e.g. a sensor fails to record a value due to a power blip

- values for some variables may be *missing systematically*: the probability of value being missing depends on the value
    - e.g. a medical test result is missing because a doctor was fairly sure of a diagnosis given earlier test results
    - e.g. the graded exams that go missing on the way home from school are those with poor scores

Missing data

- hidden variables; values *missing at random*
  - these are the cases we'll focus on
  - one solution: try impute the values

- values *missing systematically*
  - may be sensible to represent "*missing*" as an explicit feature value

Imputing missing data with EM

Given:

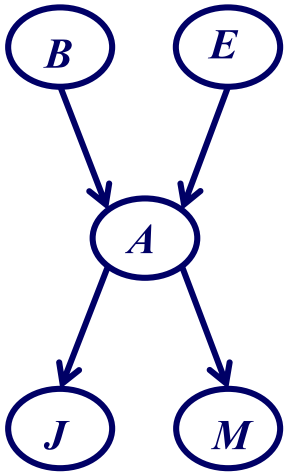
- data set with some missing values
- model structure, initial model parameters

Repeat until convergence

- *Expectation* (E) step: using current model, compute expectation over missing values
- *Maximization* (M) step: update model parameters with those that maximize probability of the data (MLE or MAP)

example: EM for parameter learning

suppose we're given the following initial BN and training set



P(B)
0.1

P(E)
0.2

B	E	P(A)
t	t	0.9
t	f	0.6
f	t	0.3
f	f	0.2

A	P(J)
t	0.9
f	0.2

A	P(M)
t	0.8
f	0.1

B	E	A	J	M
f	f	?	f	f
f	f	?	t	f
t	f	?	t	t
f	f	?	f	t
f	t	?	t	f
f	f	?	f	t
t	t	?	t	t
f	f	?	f	f
f	f	?	t	f
f	f	?	f	t

example: E-step

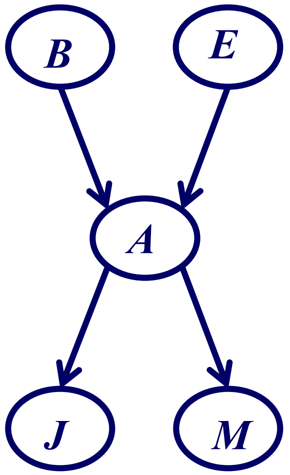
$$P(a \mid \neg b, \neg e, \neg j, \neg m)$$

$$P(\neg a \mid \neg b, \neg e, \neg j, \neg m)$$

B	*E*	*A*	*J*	*M*
f	f	t: 0.0069 f: 0.9931	f	f
f	f	t:0.2 f:0.8	t	f
t	f	t:0.98 f: 0.02	t	t
f	f	t: 0.2 f: 0.8	f	t
f	t	t: 0.3 f: 0.7	t	f
f	f	t:0.2 f: 0.8	f	t
t	t	t: 0.997 f: 0.003	t	t
f	f	t: 0.0069 f: 0.9931	f	f
f	f	t:0.2 f: 0.8	t	f
f	f	t: 0.2 f: 0.8	f	t

P(B)
0.1

P(E)
0.2



B	***E***	***P(A)***
t	**t**	**0.9**
t	**f**	**0.6**
f	**t**	**0.3**
f	**f**	**0.2**

A	***P(J)***
t	**0.9**
f	**0.2**

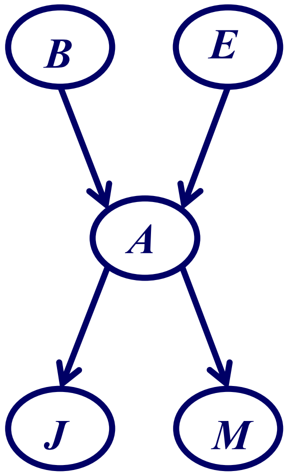
A	***P(M)***
t	**0.8**
f	**0.1**

example: E-step

P(B)
0.1

P(E)
0.2

B	E	P(A)
t	t	0.9
t	f	0.6
f	t	0.3
f	f	0.2



A	P(J)
t	0.9
f	0.2

A	P(M)
t	0.8
f	0.1

$$P(a \mid \neg b, \neg e, \neg j, \neg m)$$
$$= \frac{P(a, \neg b, \neg e, \neg j, \neg m)}{P(a, \neg b, \neg e, \neg j, \neg m) + P(\neg a, \neg b, \neg e, \neg j, \neg m)}$$
$$= \frac{0.9 \times 0.8 \times 0.2 \times 0.1 \times 0.2}{0.9 \times 0.8 \times 0.2 \times 0.1 \times 0.2 + 0.9 \times 0.8 \times 0.8 \times 0.8 \times 0.9}$$
$$= \frac{0.00288}{0.4176} = 0.0069$$

$$P(a \mid \neg b, \neg e, j, \neg m)$$
$$= \frac{P(\neg b, \neg e, a, j, \neg m)}{P(\neg b, \neg e, a, j, \neg m) + P(\neg b, \neg e, \neg a, j, \neg m)}$$
$$= \frac{0.9 \times 0.8 \times 0.2 \times 0.9 \times 0.2}{0.9 \times 0.8 \times 0.2 \times 0.9 \times 0.2 + 0.9 \times 0.8 \times 0.8 \times 0.2 \times 0.9}$$
$$= \frac{0.02592}{0.1296} = 0.2$$

example: M-step

re-estimate probabilities using expected counts

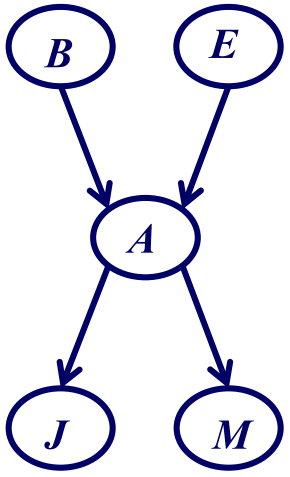
$$P(a \mid b, e) = \frac{E\#(a \wedge b \wedge e)}{E\#(b \wedge e)}$$

$$P(a \mid b, e) = \frac{0.997}{1}$$

$$P(a \mid b, \neg e) = \frac{0.98}{1}$$

$$P(a \mid \neg b, e) = \frac{0.3}{1}$$

$$P(a \mid \neg b, \neg e) = \frac{0.0069 + 0.2 + 0.2 + 0.2 + 0.0069 + 0.2 + 0.2}{7}$$



B	*E*	*P(A)*
t	t	0.997
t	f	0.98
f	t	0.3
f	f	0.145

re-estimate probabilities for $P(J \mid A)$ and $P(M \mid A)$ in same way

B	*E*	*A*	*J*	*M*
f	f	t: 0.0069 f: 0.9931	f	f
f	f	t:0.2 f:0.8	t	f
t	f	t:0.98 f: 0.02	t	t
f	f	t: 0.2 f: 0.8	f	t
f	t	t: 0.3 f: 0.7	t	f
f	f	t:0.2 f: 0.8	f	t
t	t	t: 0.997 f: 0.003	t	t
f	f	t: 0.0069 f: 0.9931	f	f
f	f	t:0.2 f: 0.8	t	f
f	f	t: 0.2 f: 0.8	f	t

example: M-step

re-estimate probabilities using expected counts

$$P(j \mid a) = \frac{E\#(a \wedge j)}{E\#(a)}$$

$$P(j \mid a) = \frac{0.2 + 0.98 + 0.3 + 0.997 + 0.2}{0.0069 + 0.2 + 0.98 + 0.2 + 0.3 + 0.2 + 0.997 + 0.0069 + 0.2 + 0.2}$$

$$P(j \mid \neg a) = \frac{0.8 + 0.02 + 0.7 + 0.003 + 0.8}{0.9931 + 0.8 + 0.02 + 0.8 + 0.7 + 0.8 + 0.003 + 0.9931 + 0.8 + 0.8}$$

B	E	A	J	M
f	f	t: 0.0069 f: 0.9931	f	f
f	f	t:0.2 f:0.8	t	f
t	f	t:0.98 f: 0.02	t	t
f	f	t: 0.2 f: 0.8	f	t
f	t	t: 0.3 f: 0.7	t	f
f	f	t:0.2 f: 0.8	f	t
t	t	t: 0.997 f: 0.003	t	t
f	f	t: 0.0069 f: 0.9931	f	f
f	f	t:0.2 f: 0.8	t	f
f	f	t: 0.2 f: 0.8	f	t

Convergence of EM

- E and M steps are iterated until probabilities converge
- will converge to a maximum in the data likelihood (MLE or MAP)
- the maximum may be a local optimum, however
- the optimum found depends on starting conditions (initial estimated probability parameters)

Learning structure + parameters

- number of structures is superexponential in the number of variables
- finding optimal structure is NP-complete problem
- two common options:
  - search very restricted space of possible structures (e.g. networks with tree DAGs)
  - use heuristic search (e.g. sparse candidate)

The Chow-Liu algorithm

- learns a BN with a tree structure that maximizes the likelihood of the training data
- algorithm
  1. compute weight $I(X_i, X_j)$ of each possible edge (X_i, X_j)
  2. find maximum weight spanning tree (MST)
  3. assign edge directions in MST

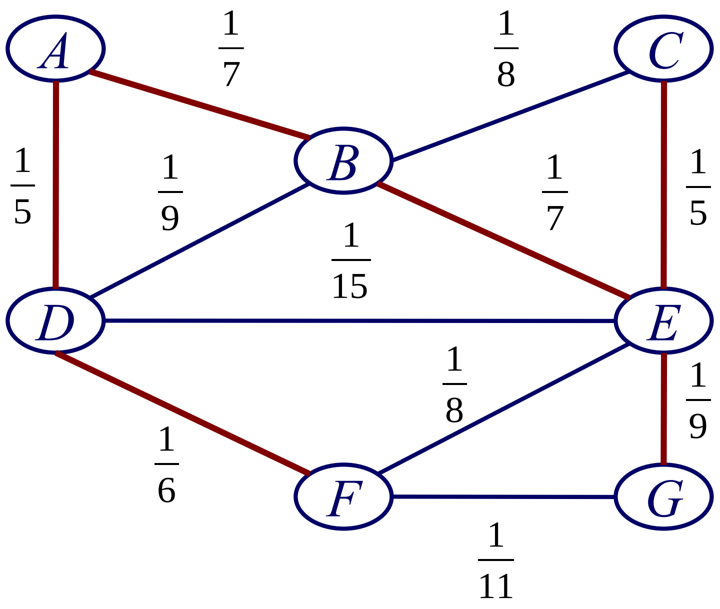
The Chow-Liu algorithm

1. use mutual information to calculate edge weights

$$I(X,Y) = \sum_{x \in \text{values}(X)} \sum_{y \in \text{values}(Y)} P(x,y) \log_2 \frac{P(x,y)}{P(x)P(y)}$$

The Chow-Liu algorithm

2. find maximum weight spanning tree: a maximal-weight tree that connects all vertices in a graph



Prim’s algorithm for finding an MST

given: graph with vertices V and edges E

```
$V_{new} \leftarrow \{ v \}$  where $v$ is an arbitrary vertex from $V$
$E_{new} \leftarrow \{ \}$
repeat until $V_{new} = V$
{
    choose an edge $(u, v)$ in $E$ with max weight where $u$ is in $V_{new}$ and $v$ is not
    add $v$ to $V_{new}$ and $(u, v)$ to $E_{new}$
}
return $V_{new}$ and $E_{new}$ which represent an MST
```

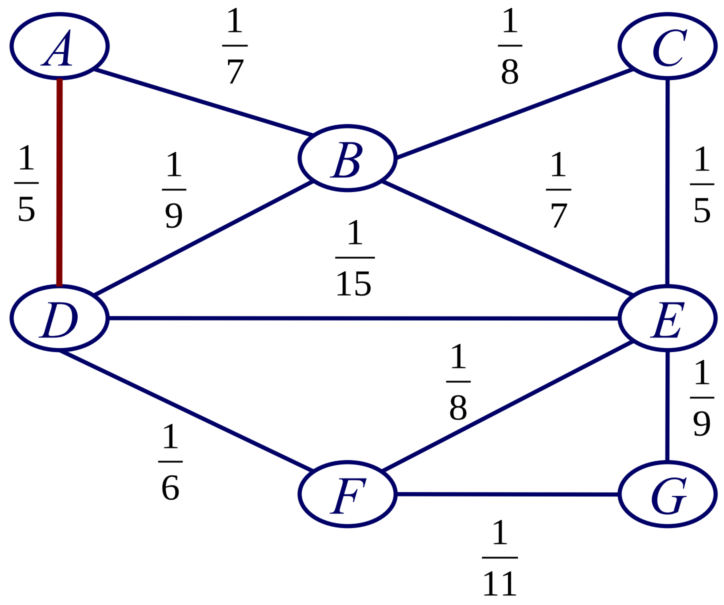
Kruskal's algorithm for finding an MST

given: graph with vertices V and edges E

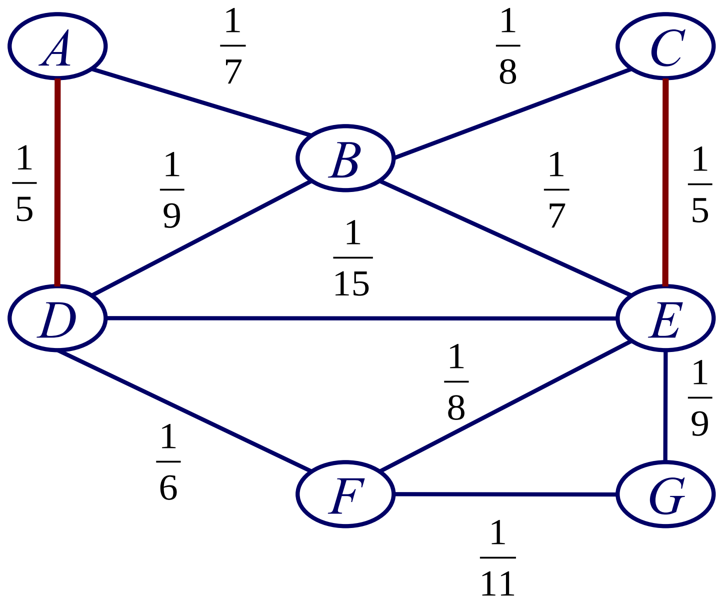
```
$E_{new} \leftarrow \{\ \}$
for each $(u, v)$ in $E$ ordered by weight (from high to low)
{
    remove $(u, v)$ from $E$
    if adding $(u, v)$ to $E_{new}$ does not create a cycle
        add $(u, v)$ to $E_{new}$
}
return $V$ and $E_{new}$ which represent an MST
```

Finding MST in Chow-Liu

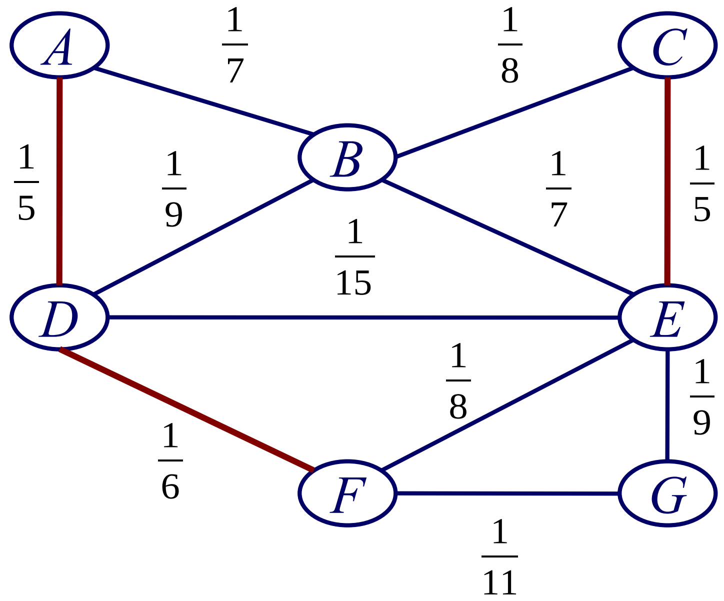
i.



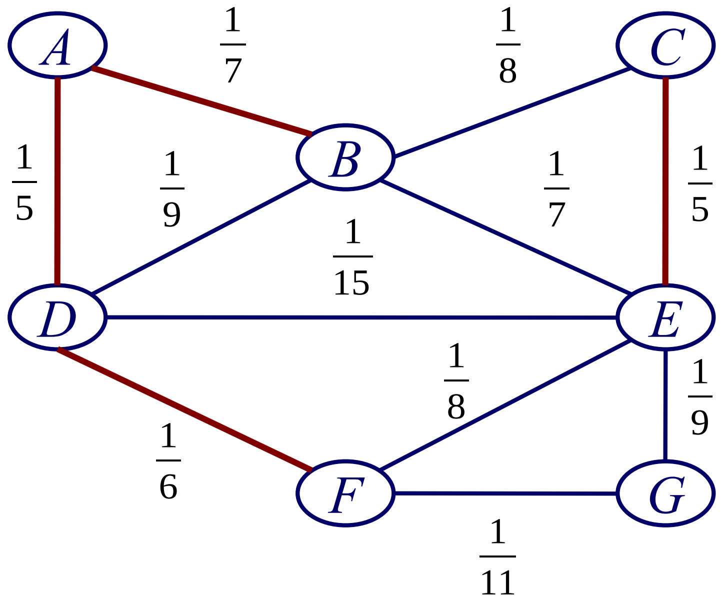
ii.



iii.

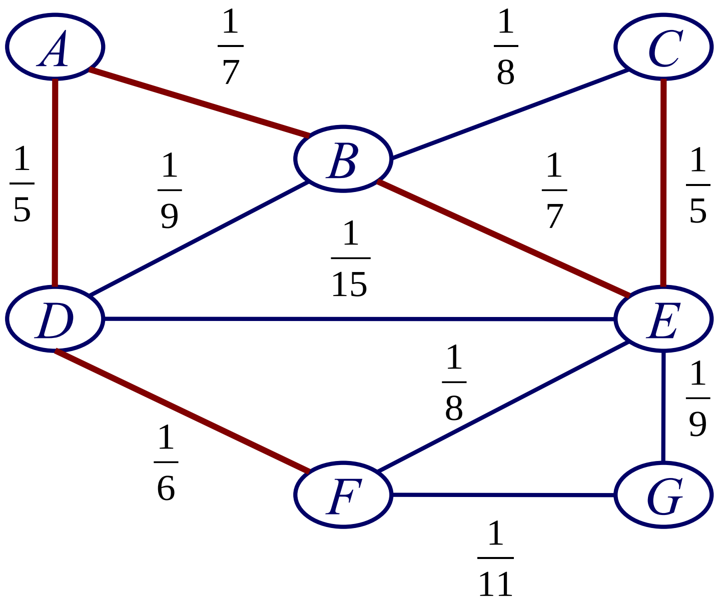


iv.

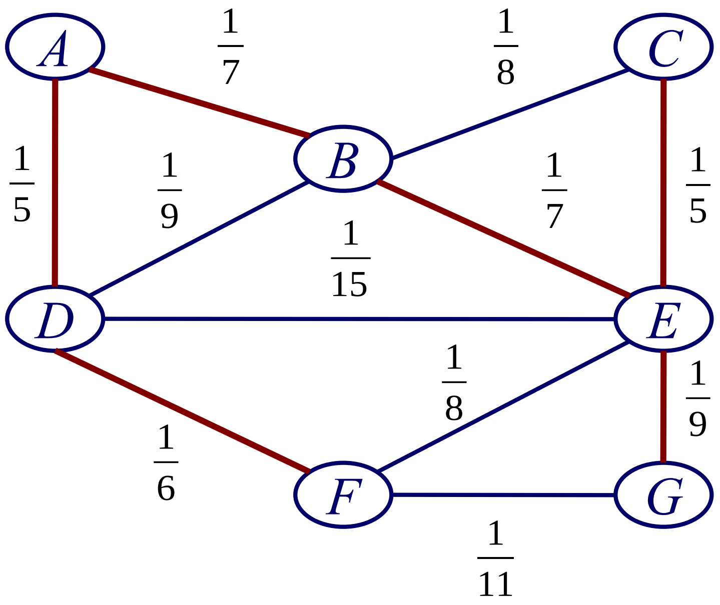


Finding MST in Chow-Liu

v.

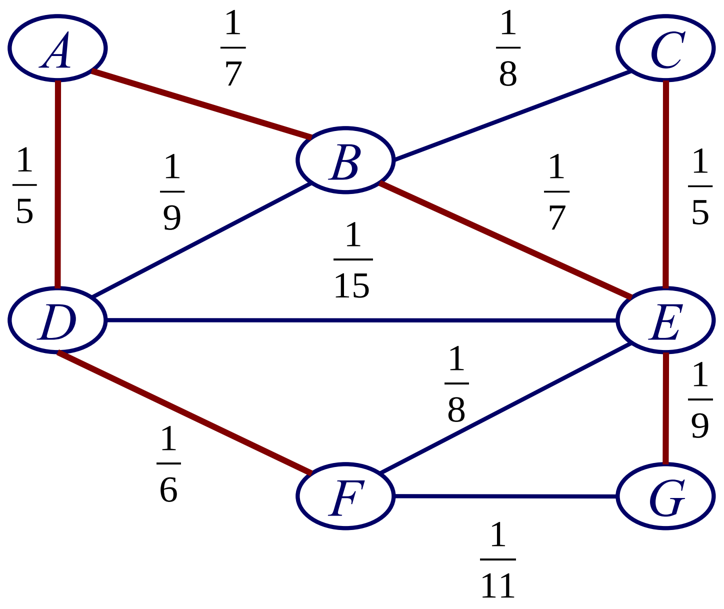


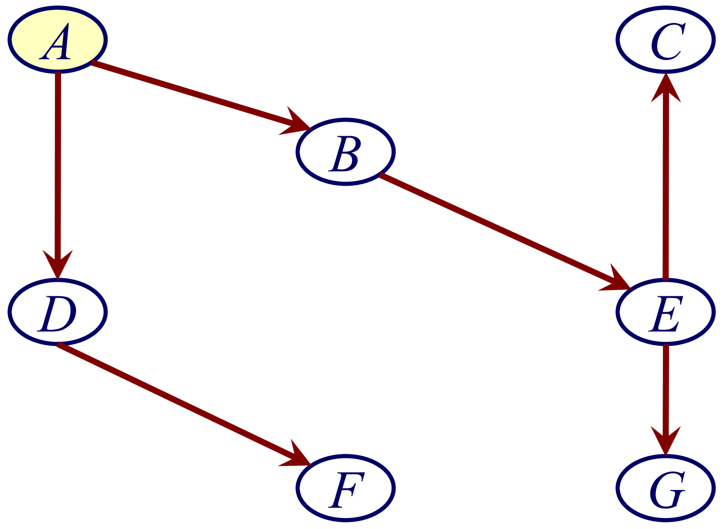
vi.



Returning directed graph in Chow-Liu

3. pick a node for the root, and assign edge directions





The Chow-Liu algorithm

- How do we know that Chow-Liu will find a tree that maximizes the data likelihood?
- Two key questions:
  - Why can we represent data likelihood as sum of $I(X;Y)$ over edges?
  - Why can we pick any direction for edges in the tree?

Why Chow-Liu maximizes likelihood (for a tree)

data likelihood given directed edges

$$\log_2 P(D \mid G, \theta_G) = \sum_{d \in D} \sum_i \log_2 P(x_i^{(d)} \mid Parents(X_i))$$

$$= |D| \sum_i (I(X_i, Parents(X_i)) - H(X_i))$$

we're interested in finding the graph G that maximizes this

$$\arg\max_G \log_2 P(D \mid G, \theta_G) = \arg\max_G \sum_i I(X_i, Parents(X_i))$$

if we assume a tree, each node has at most one parent

$$\arg\max_G \log_2 P(D \mid G, \theta_G) = \arg\max_G \sum_{(X_i, X_j) \in \text{edges}} I(X_i, X_j)$$

edge directions don't matter for likelihood, because MI is symmetric

$$I(X_i, X_j) = I(X_j, X_i)$$