1 Intro to NP-Completeness

Assume that I have problems $A, B$. We say that that $B$ is as hard as $A$, if there exists a polynomial reduction from $A$ to $B$. Meaning that if I can solve $B$ then I can solve $A$. We write $A \leq_p B$. One known NP-complete problem is called "Set-Splitting" and is the following:

**Set-Splitting (SS):**

- **Given:** $S$, a collection of subsets $C = \{C_i | C_i \subset S\}$.
- **Question:** Does there exists $S_1, S_2$ with $S_1 \cap S_2 = \emptyset$, such that $S_1 \cup S_2 = S$, and for all $i$, it holds $C_i \not\subset S_1$ and $C_i \not\subset S_2$.

Note that the problem is still hard assuming $|C| = O(|S|)$.

In the section below, we are going to reduce the "Set-Splitting" to the training a 3-node NN. This was proved in [1].

2 Training a 3-Node Network

![Diagram of a 3-Node Network](image)

Figure 1: The 3-Node network
Let $a = [a_1, \ldots, a_n] \in \mathbb{R}^n$ and $a_0 \in \mathbb{R}$. We define the following threshold function:

$$f_i(z) = \begin{cases} 
1 & \text{if } a \cdot z > a_0 \\
-1 & \text{o.w.}
\end{cases}$$

This is equivalent to $f_i(z) = \text{sign}(a \cdot z - a_0)$. The main question is the following:

**Question:** Given a set of $O(n)$ examples $(x, y) \in \{0, 1\}^n \times \{-1\}$. Do there exists, $f_1, f_2, f_3$ such that the 3-node network has training error 0?

In fact, we are going to show that this problem is hard and in fact it is NP-Complete, by showing a reduction from Set-Splitting problem. Hence, we show the following:

**Theorem 1.** Training 3-node NN is NP-Complete.

**Proof.** First, we provide a geometric intuition for this problem: Each point is a point of the $n$-dimensional hypercube. The two functions $f_1, f_2$ are linear thresholds functions, therefore, each one define a hyperplane. Therefore, if they are not parallel, they divide the space into four quadrants. Because the $f_3$ is a linear threshold, it can distinguish between points on different quadrants. So, the problem of training a 3-node, is equivalent to the following problem:

Given a set of labeled points in the $n$-dimensional hypercube does there exists:

1. **Case 1:** A simple plane separates $\pm 1$.
2. **Case 2:** Two planes such that either one quadrant contains all positive labels $(+1)$ and no negative points, or one quadrant contains all negative labels $(-1)$ and no positive points.

We are going to show that case 2 is the hard one, which means that this problem is NP-complete.

**Problem 2LCPBE:** Given $n$-labeled points. Do there exist planes $f_1, f_2$ such that the quadrant with both positive predictions contains all positive points and no negative labeled points?

We are going to reduce the problem of Set-Splitting to 2LCPBE. Given an instance of Set-Splitting: $S = \{s_1, s_2, \ldots, s_n\}$, $C = \{C_1, C_2, \ldots\}$ and $\{C_j \subseteq S\}$, we are going to convert it to the following instance of 2LCPBE:

- Let the origin: $(0,0,\ldots,0)$ have the label $+$.  
- for each $s_i$, we make a point $p_i = (0,\ldots,1,\ldots,0)$, where the 1 is in the $i$-th position, and we label it $-$.  
- for all $C_j = \{s_{j1},\ldots,s_{jk}\}$, we put at the point that has 1 at the positions $j1,j2,\ldots,jk$ and the $+$ label, that point is $p_{j1}+p_{j2},\ldots,p_{jk}$.

For example consider the instance: $S = \{s_1,s_2,s_3\}$, $C_1 = \{s_1,s_2\}$ $C_2 = \{s_2,s_3\}$. We have $[(0,0,0),1]$ and $[(1,0,0),-1],[0,1,0),-1],[0,0,1),-1]$ and $[(1,1,0),1],[0,1,1),1]$.  

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Lemma 2. The instance of $SS$ has a solution is equivalent to constructed instance of $2LCPBE$ has.

Proof. For the first direction. Given $S_1, S_2$ from the solution of set splitting, we consider the following:

Consider the hyperplanes: $P_1, P_2$ with the following form: $P_j : a_1 x_1 + \ldots + a_n x_n + 1/2 = 0,$ where

$$a_i = \begin{cases} -1 & \text{if } s_i \in S_j \\ n & \text{ow} \end{cases}$$

You can see that the following hold:

- $P_j$ predicts $+$ for $(0, 0, \ldots, 0)$
- $P_j$ predicts $+$ for training point with $+$
- $P_j$ predicts $-$ for $p_i$ if $s_i \in S_i.$

Therefore, the intersection (the quadrant) of hyperplanes: $P_1 \geq 0, P_2 \geq 0,$ contains all the points with $+$ and no point with $-$.

Let $S_1$ (resp. $S_2$) be the set that contains that only in $P_1$ (resp. $P_2$) get $-$. Place the rest of the points that both planes separates with $-$ arbitrary in $S_1$ or $S_2$. $S_1 \cup S_2 = S$ as all the points are either in $S_1$ or $S_2$.

Let $C_j = \{s_{j1}, \ldots, s_{jk}\}$, it remains to show that $C_j \not\subset S_1, S_2$. $P_1$ predicts positive for $p_{j1} + \ldots + p_{jk}$ if $c_j \subset S_1$ but then this points would not be in one quadrant with only positive points which contradicts the assumption of $2LCPBE$. Similarly for $P_2$ and $S_2.$

Now we have shown that the training 3-node is NP-complete if one quadrant contains all the positive points, so the $f_3$ should be the AND function between $f_1$ and $f_2$. Now, we will add some more points to make the output to always require that conditions. We extend the dimension of our points to $n + 3$ and put $0$ in the new components of the previous points. This is left as homework; you can also refer to the reference.

References