CS 540 Introduction to Artificial Intelligence
Games II

Yudong Chen
University of Wisconsin-Madison

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Announcements

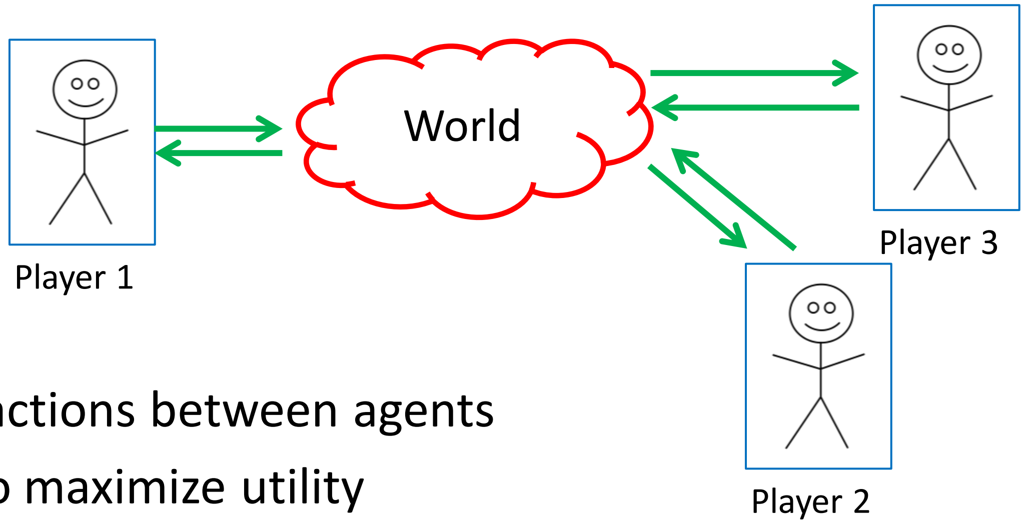
- **Homeworks**:
  - HW 9 deadline extended to next Tuesday (Dec 7)
  - HW 10 to be released on Thursday. Due Dec 14 Tuesday
- **Class roadmap:**
  - Today: Game II
  - Next: Reinforcement Learning

Outline

- Review of game theory basics
  - Properties, mathematical setup, simultaneous games
- Sequential games
  - Game trees, minimax, search approaches
- Speeding up sequential game search
  - Pruning, heuristics

Review of Games: Multiple Agents

Games setup: **multiple** agents



- Now: interactions between agents
- Still want to maximize utility
- **Strategic** decision making.

Review of Games: Properties

Let's work through **properties** of games

- **Number** of agents/players
- State & action spaces: **discrete** or **continuous**
- **Finite** or **infinite**
- **Deterministic** or **random**
- **Sum**: zero or positive or negative
- **Sequential** or **simultaneous**

Wiki

Review: Prisoner's Dilemma

Famous example from the '50s.

Two prisoners A & B. Can choose to betray the other or not.

- A and B both betray, each of them serves two years in prison
- One betrays, the other doesn't: betrayer free, other three years
- Both do not betray: one year each

Properties: **2-player, discrete, finite, deterministic, negative-sum, simultaneous**

Review: Normal Form

Mathematical description of simult. games. Has:

- n players $\{1,2,\ldots,n\}$
- Player i strategy a_i from A_i. **All**: $a = (a_1, a_2, \ldots, a_n)$
- Player i gets rewards $u_i(a)$ for any outcome
  - **Note**: reward depends on other players!

- Setting: all of these spaces, rewards are **known**

Review: Example of Normal Form

Ex: Prisoner's Dilemma

Player 2 / Player 1	*Stay silent*	*Betray*
Stay silent	−1, −1	−3, 0
Betray	0, −3	−2, −2

- 2 players, 2 actions: yields 2x2 matrix
- Strategies: {Stay silent, betray} (i.e, binary)
- Rewards: {0,-1,-2,-3}

Review: Dominant Strategies

Let's analyze such games. Some strategies are better

- Dominant strategy: if a_i better than a_i' *regardless* of what other players do, a_i is **dominant**
- I.e.,

$$u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}) \forall a_i' \neq a_i \text{ and } \forall a_{-i}$$

All of the other entries
of *a* excluding *i*

- Doesn't always exist!

Review: Equilibrium

$=(a_1^*, \cdots, a_n^*)$

$a*$ is an equilibrium if all the players do not have an incentive to **unilaterally deviate**

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*) \quad \forall a_i \in A_i \qquad \cancel{\forall a_{-i}}$$

- All players dominant strategies -> equilibrium
- Converse doesn't hold (don't need dominant strategies to get an equilibrium)

Review: Pure and Mixed Strategies

So far, all our strategies are deterministic: "**pure**"

- Take a particular action, no randomness

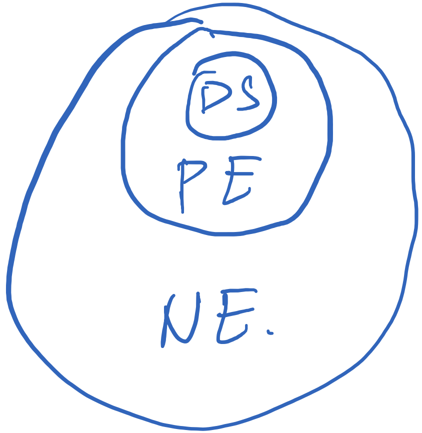
Can also randomize actions: "**mixed**"

- Assign probabilities x_i to each action

$$x_i(a_i), \text{ where } \sum_{a_i \in A_i} x_i(a_i) = 1, x_i(a_i) \geq 0$$

- Note: have to now consider **expected rewards**

Review: Nash Equilibrium



Consider the mixed strategy $x^* = (x_1^*, \ldots, x_n^*)$

- This is a **Nash equilibrium** if

$$u_i(x_i^*, x_{-1}^*) \geq u_i(x_i, x_{-i}^*) \quad \forall x_i \in \Delta_{A_i}, \forall i \in \{1, \ldots, n\}$$

Better than doing anything else, "**best response**"

Space of probability distributions

$(x_1^*, \cdots, x_n^*)$ is NE $\Longleftrightarrow$ $(x_1^*, \cdots, x_n^*)$ are best response to each other

- Intuition: nobody can **increase expected reward** by changing only their own strategy. A type of solution!

Break & Quiz

Q 1.1: Which of the following is **false**

- A. The set of mixed strategies includes pure strategies.
- B. A game can be simultaneous but have imperfect information.
- C. A game may not have any dominating strategies.
- D. All finite two player games have pure Nash equilibria.

Break & Quiz

Q 1.1: Which of the following is **false**

- A. The set of mixed strategies includes pure strategies.
- B. A game can be simultaneous but have imperfect information.
- C. A game may not have any dominating strategies.
- **D. All finite two player games have pure Nash equilibria.**

Break & Quiz

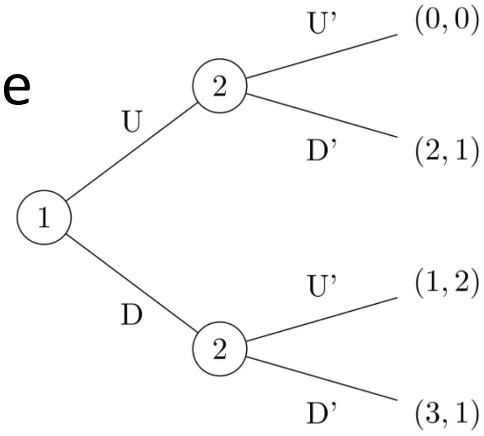
Q 1.1: Which of the following is **false**

- A. The set of mixed strategies includes pure strategies. (Yes: deterministic is a subset of random---such strategies have all the probability mass on one outcome.)
- B. A game can be simultaneous but have imperfect information. (Example: RPS. There're sequential games with imperfect info. as well, e.g., Battleship.)
- C. A game may not have any dominating strategies. (RPS).
- **D. All finite two player games have pure Nash equilibria.** (This is false---they all have mixed equilibria, but not necessarily pure.)

Sequential Games

More complex games with multiple moves

- Instead of normal form, **extensive form**
- Represent with a **tree**
- Find strategies: perform search over the tree



Wiki

- Can still look for Nash equilibrium
  - Or, other criteria like **minimax**

II-Nim: Example Sequential Game

2 piles of sticks, each with 2 sticks.

- Each player takes one or more sticks from one of the piles
- Take last stick: lose

(ii, ii)

- Two players: Max and Min
- If Max wins, the score is **+1**; otherwise **-1**
- Min's score is –Max's
- Use Max's as the score of the game

Game Trajectory

(ii, ii)

Game Trajectory

(ii, ii)

Max takes one stick from one pile

(i, ii)

Game Trajectory

(ii, ii)

Max takes one stick from one pile

(i, ii)

Min takes two sticks from the other pile

(i,-)

Game Trajectory

(ii, ii)

Max takes one stick from one pile

(i, ii)

Min takes two sticks from the other pile

(i,-)

Max takes the last stick

(-,-)

Max gets score **-1**

Game tree for II-Nim

Two players:
Max and Min

(ii ii) Max

who is to move at this state

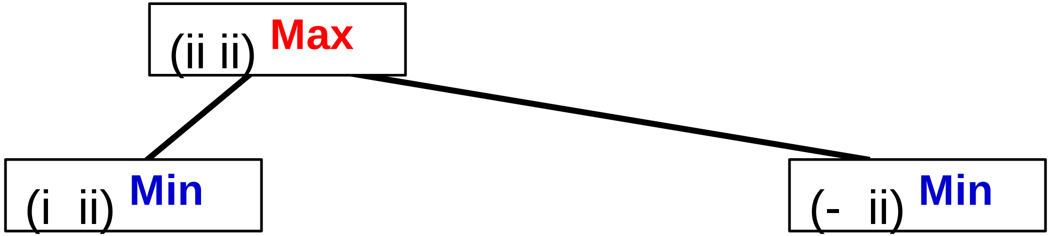
Convention: score is w.r.t. the first player Max. Min's score = – Max

Max wants the largest score
Min wants the smallest score

Game tree for II-Nim

Two players:
Max and Min

Symmetry
(i ii) = (ii i)



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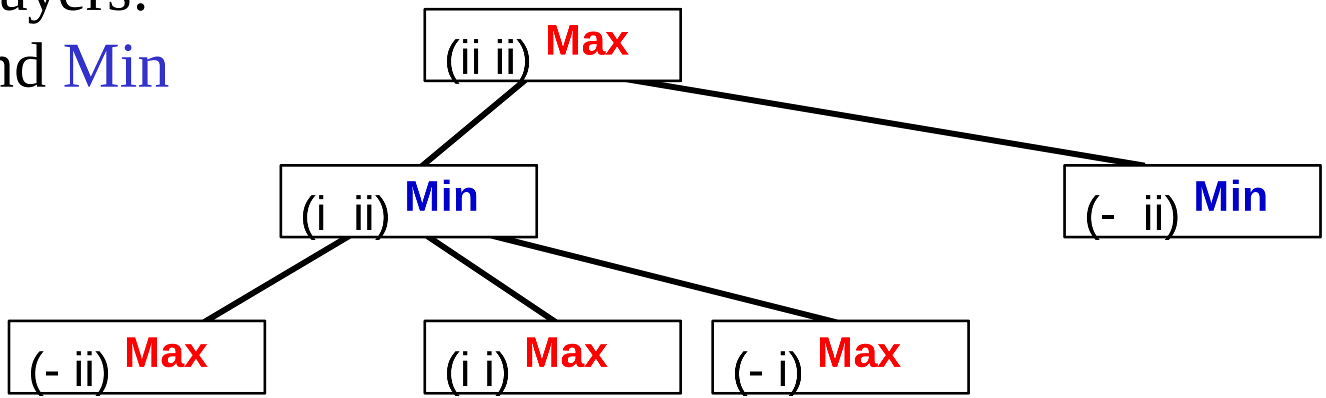
Game tree for II-Nim

Two players:
Max and Min



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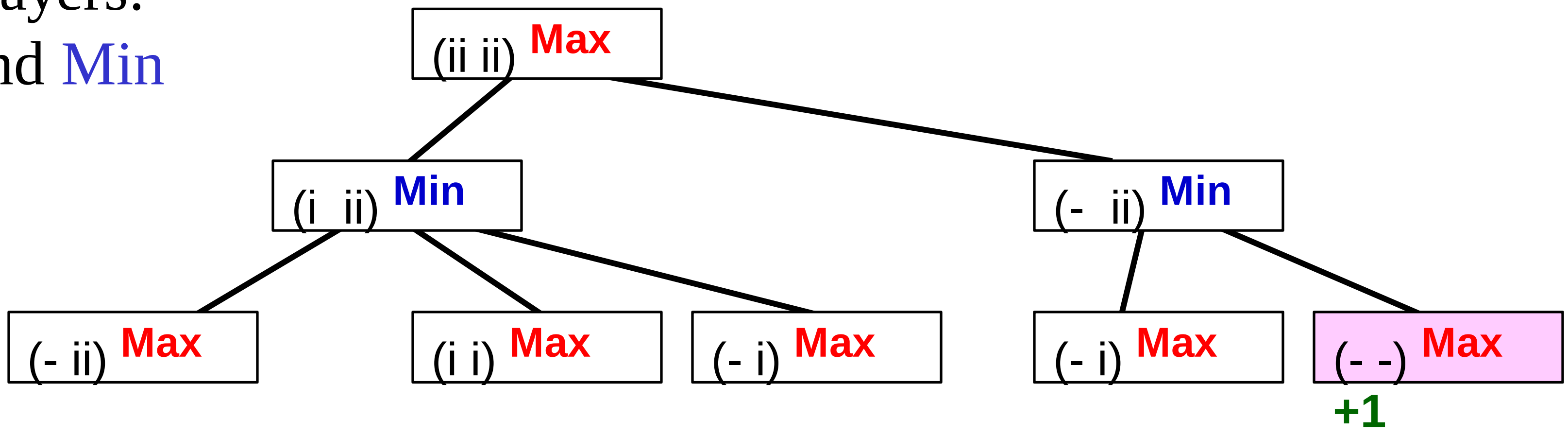

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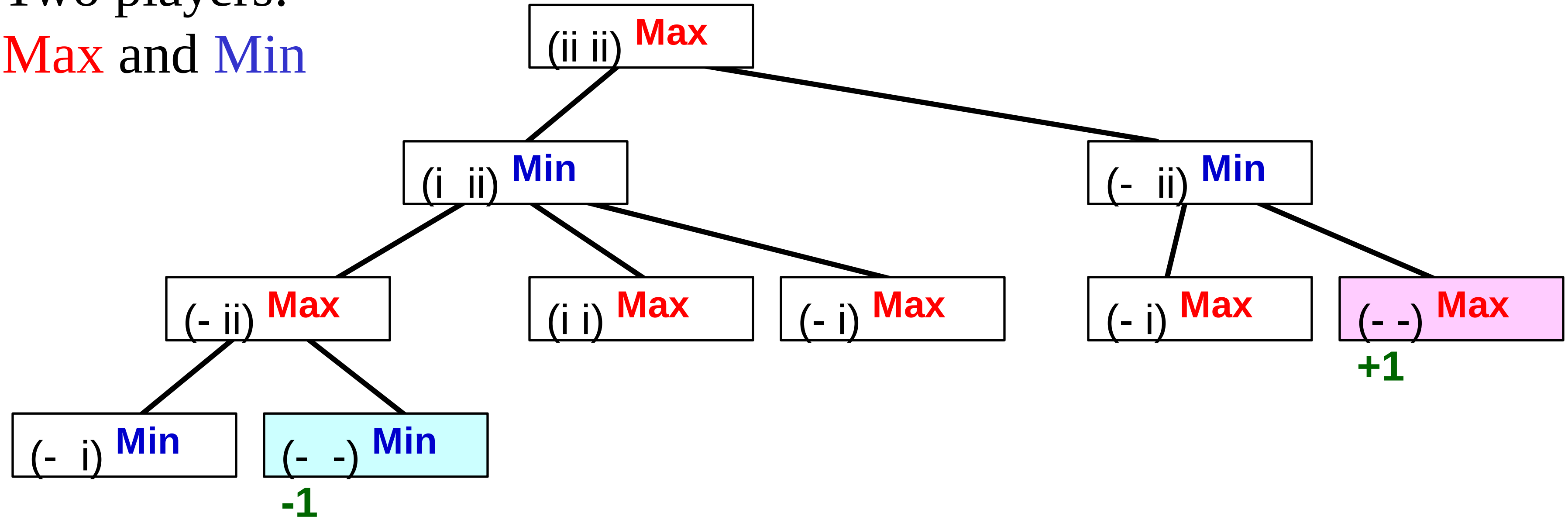

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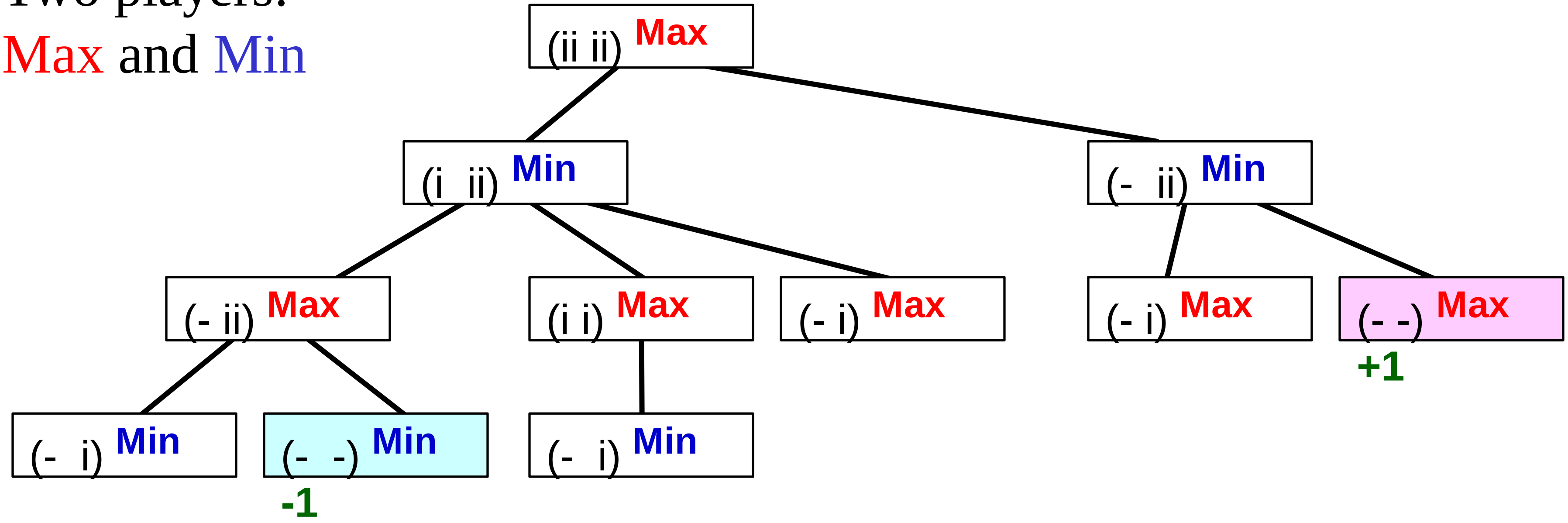

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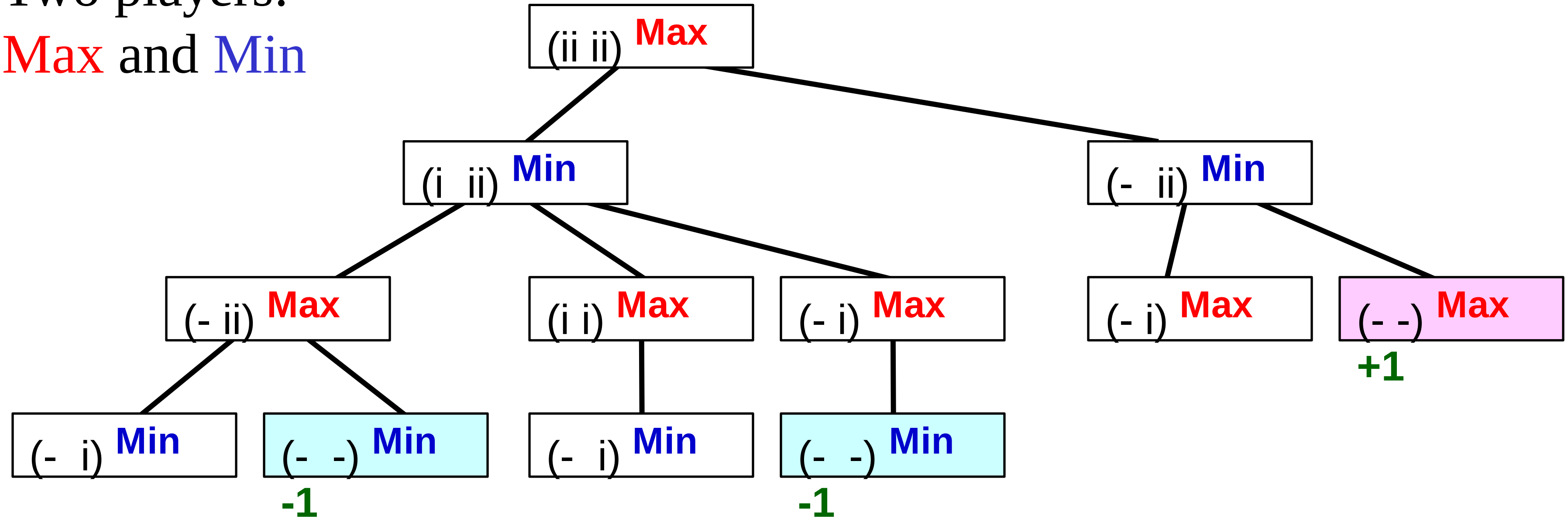

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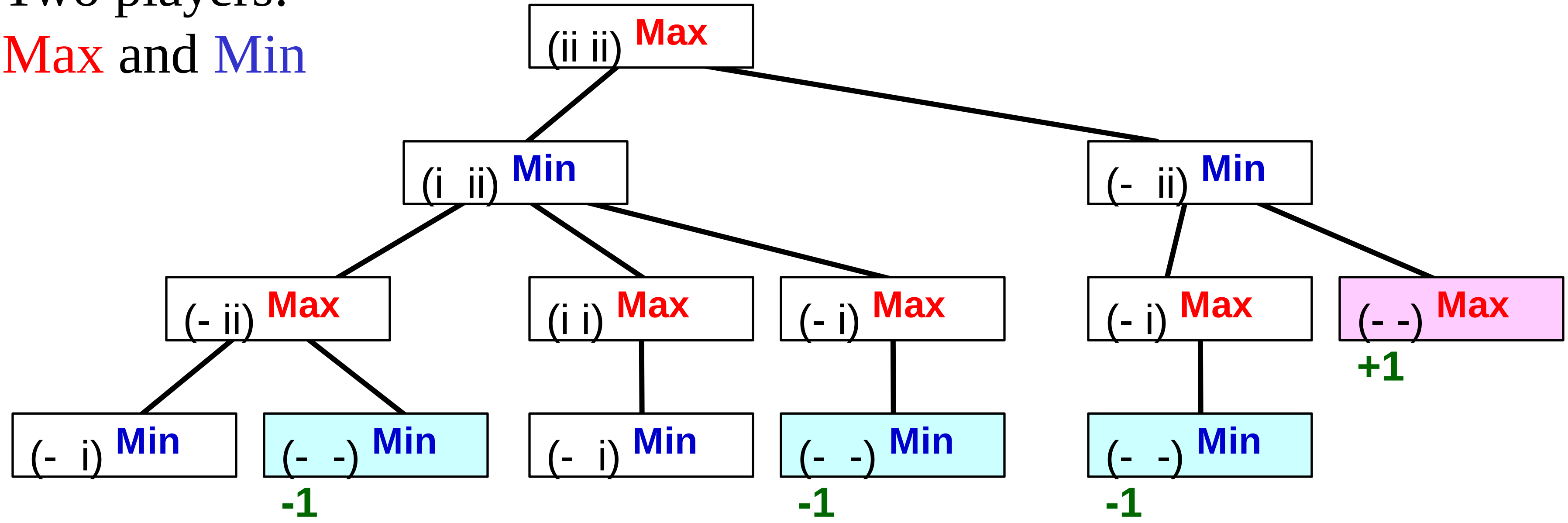

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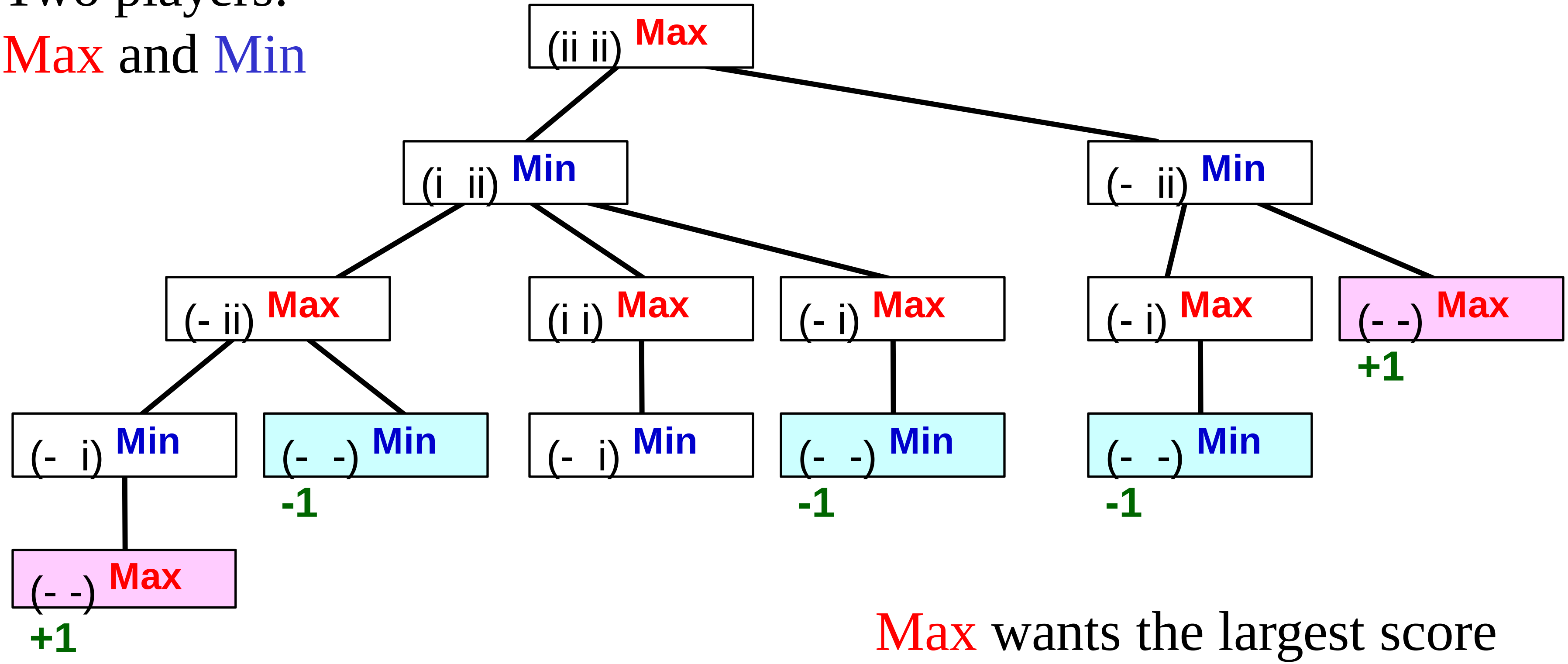

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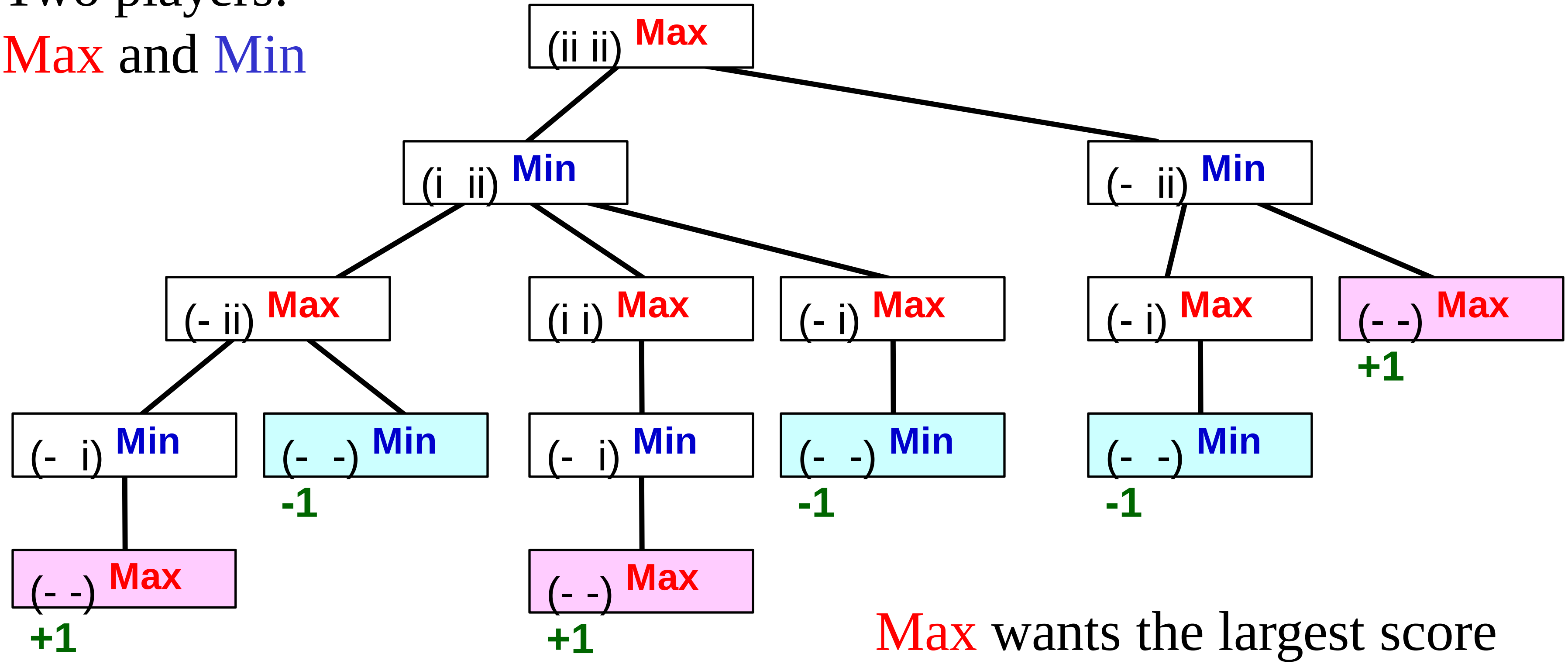

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Strategies & Rewards

sequential

Let's stick to zero-sum two-player games

- Strategies: player 1 (Max): s, player 2 (Min): t
- Player 1 (Max): reward $u(s,t)$, player 2 (Min): $-u(s,t)$

- Max goal: maximize $u(s,t)$

Min goal: minimize $u(s,t)$

$\Leftrightarrow$ maximize $-u(s,t)$

- Goal: find strategies s, t that do this.

Minimax Theorem

Famous result of von Neumann

- Says: there are strategies $s*$ and $t*$ and a value $u*$, the **minimax value** so that
  - If **Min** uses $t*$, then **Max's** reward $\leq u*$ (i.e., $\max_s u(s, t*) = u*$)
  - If **Max** uses $s*$, then Min ~~**Max's**~~ reward $\geq u*$ (i.e., $\min_t u(s*, t) = u*$)
- So: $u(s*, t*) = u*$
- Also: if game has perfect information, there are pure strategies $s*$, $t*$ that satisfy the result

Finding The Strategies

Back to our game tree

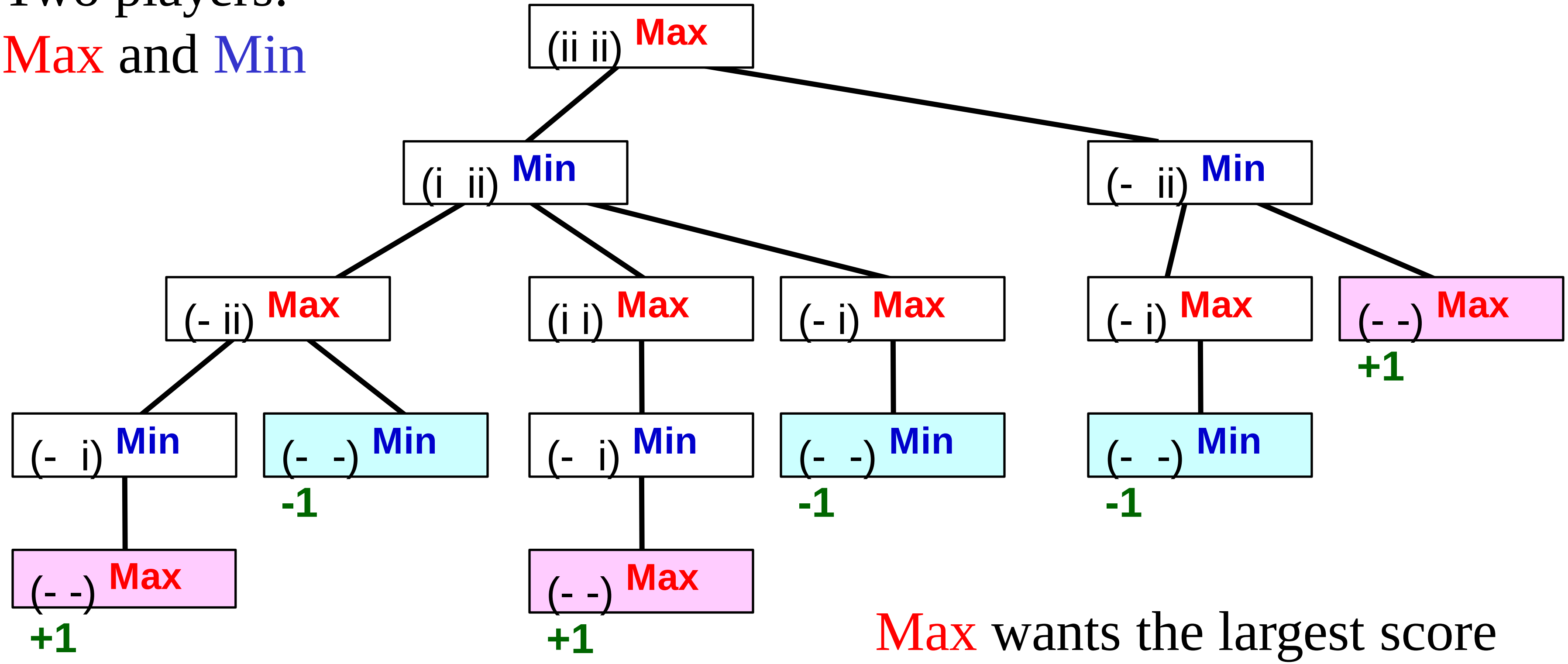
- Write down all the pure strategies (e.g., the big tree) and select the *s** and *t**

$$s^* = \arg\max_{s \in S} \min_{t \in T} u(s,t) \qquad t^* = \arg\min_{t \in T} \max_{s \in S} u(s,t)$$

- Big search, since for branching factor *b*, height *h*, need to look at ~b^h strategies

Game tree for II-Nim

Two players:
Max and Min



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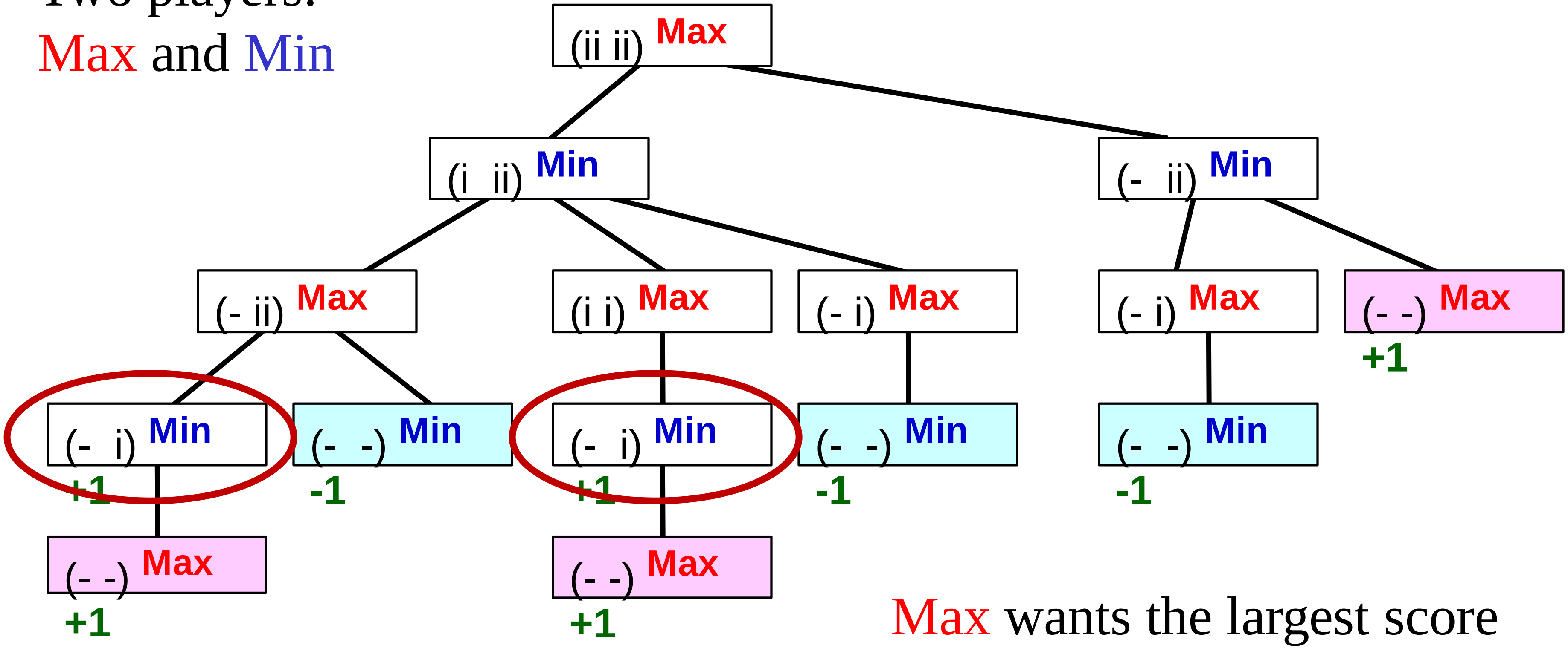

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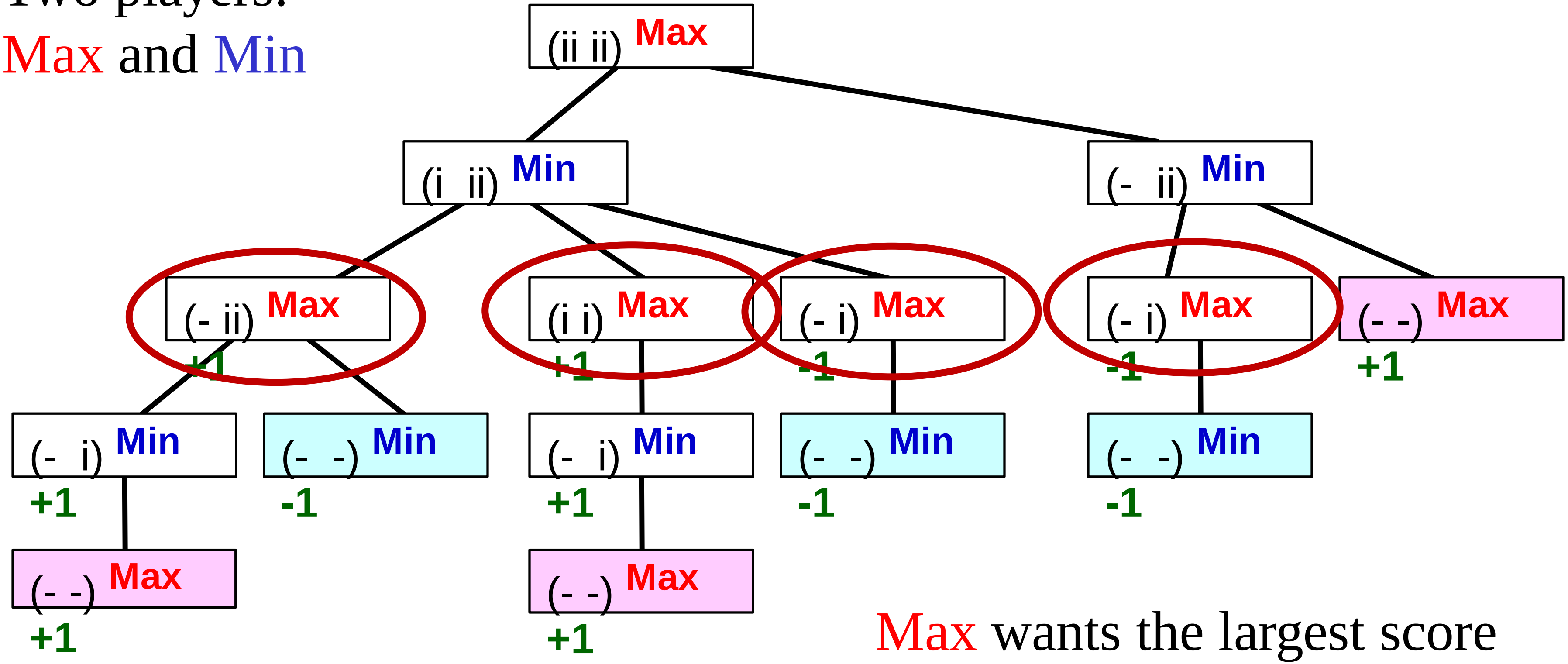

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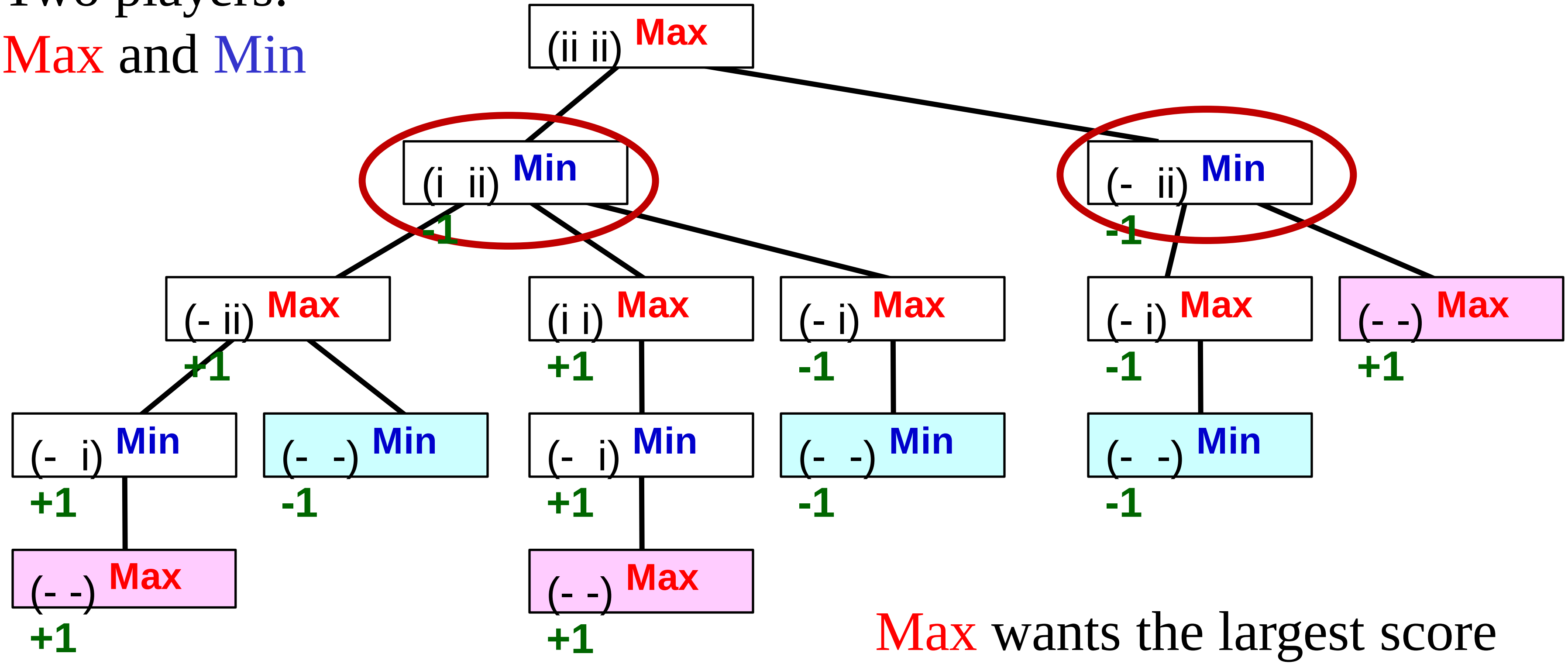

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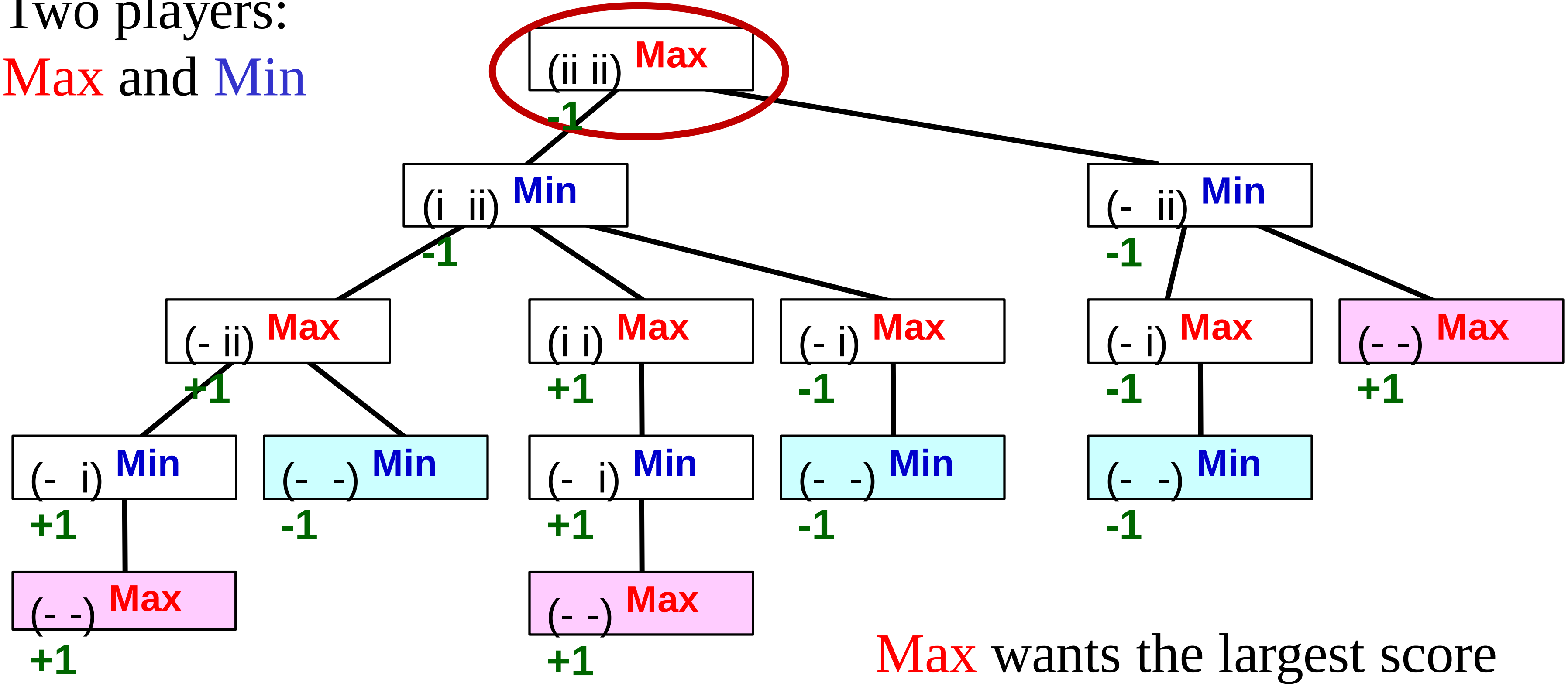

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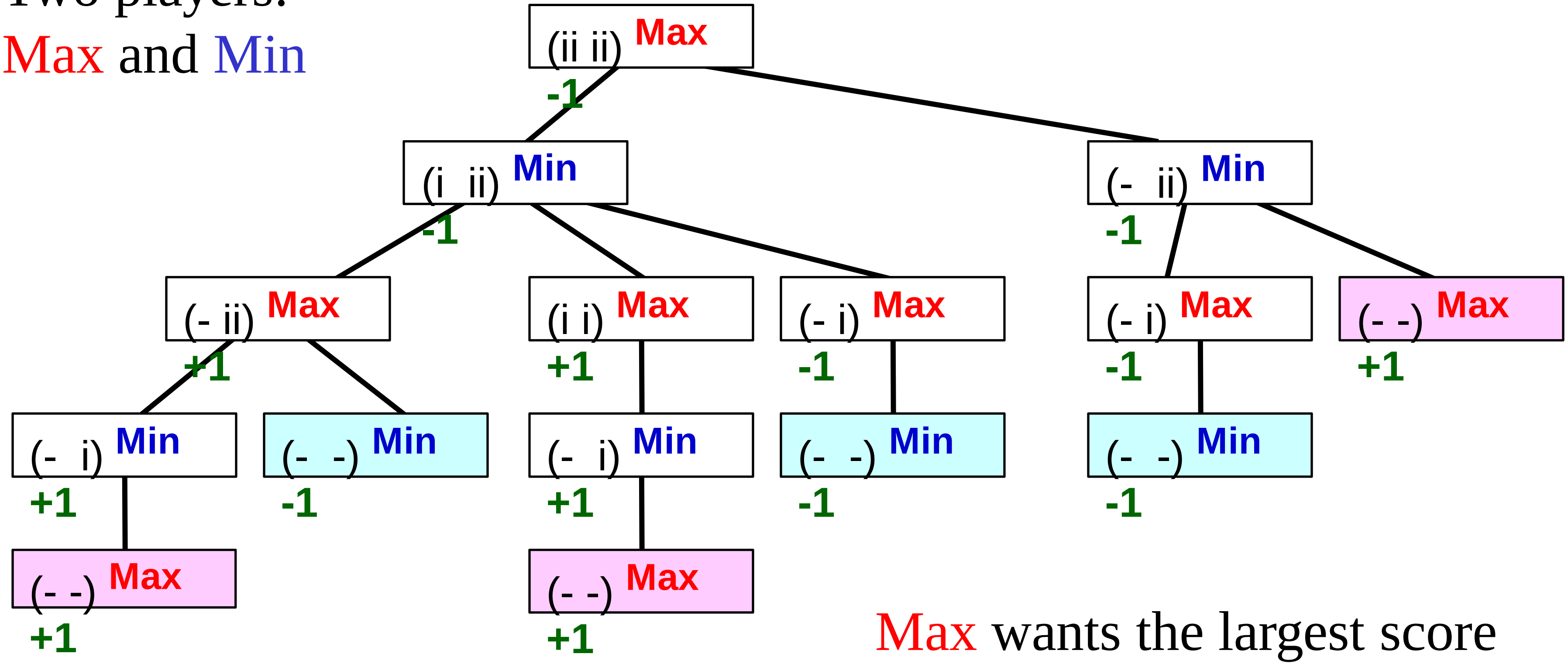

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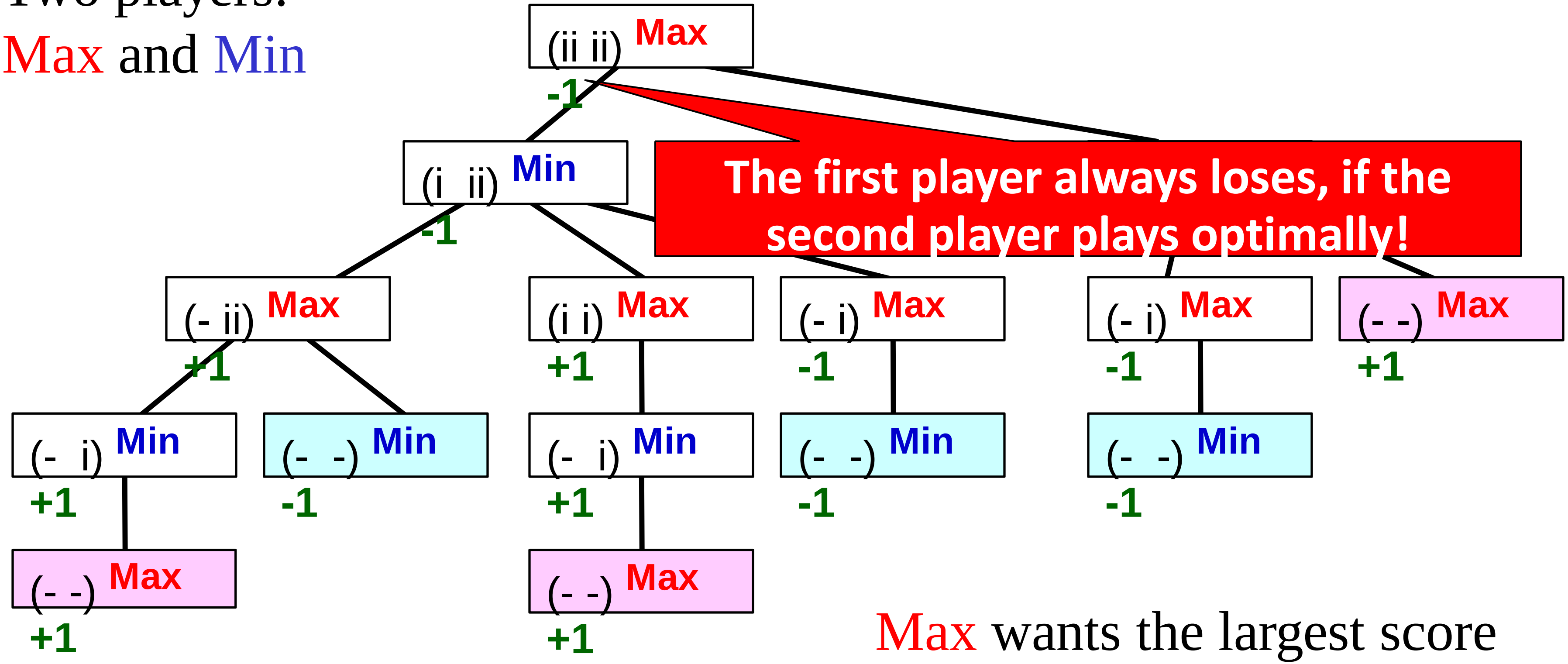

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Break & Quiz

Q 2.1: We are playing a game where Player A goes first and has 4 moves. Player B goes next and has 3 moves. Player A goes next and has 2 moves. Player B then has one move.

How many nodes are there in the minimax tree, including termination nodes (leaves)?

- A. 23
- B. 65
- C. 41
- D. 2

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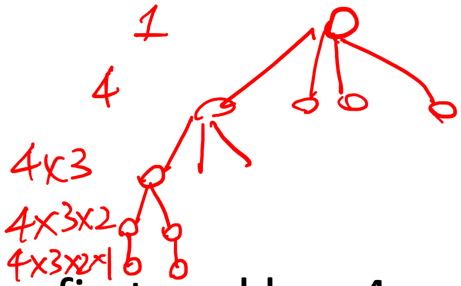

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How many nodes are there in the minimax tree, including termination nodes (leaves)?

- A. 23
- **B. 65** (1 + 4 + 4*3 + 4*3*2 + 4*3*2 = 65. Note the root and leaf nodes.)
- C. 41
- D. 2

Break & Quiz

Q 2.2: During minimax tree search, must we examine every node?

- A. Always
- B. Sometimes
- C. Never

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Break & Quiz

Q 2.2: During minimax tree search, must we examine every node?

See pruning (later)

- A. Always (No: consider layer *k*, where we take the max of all the mins of its children at layer *k+1*. If the current value of a min node at *k+1* already smaller than the current max, we don't need to continue the minimization.)
- **B. Sometimes**
- C. Never (No: the event above may simply not happen).

Our Approach So Far

We find the minimax value/strategy bottom up

- Minimax value: score of terminal node when both players play optimally
  - **Max's** turn, take max of children
  - **Min's** turn, take min of children

- Can implement this as depth-first search: **minimax algorithm**

Minimax Algorithm

```
function Max-Value(s)
inputs:
      s: current state in game, Max about to play
output: best-score (for Max) available from s

      if ( s is a terminal state )
      then return ( terminal value of s )
      else
                  α := – infinity
                  for each s’ in Succ(s)
                      α := max( α , Min-value(s’))
      return α
```

```
function Min-Value(s)
output: best-score (for Min) available from s

      if ( s is a terminal state )
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      else
                  β := infinity
                  for each s’ in Succs(s)
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      return β
```

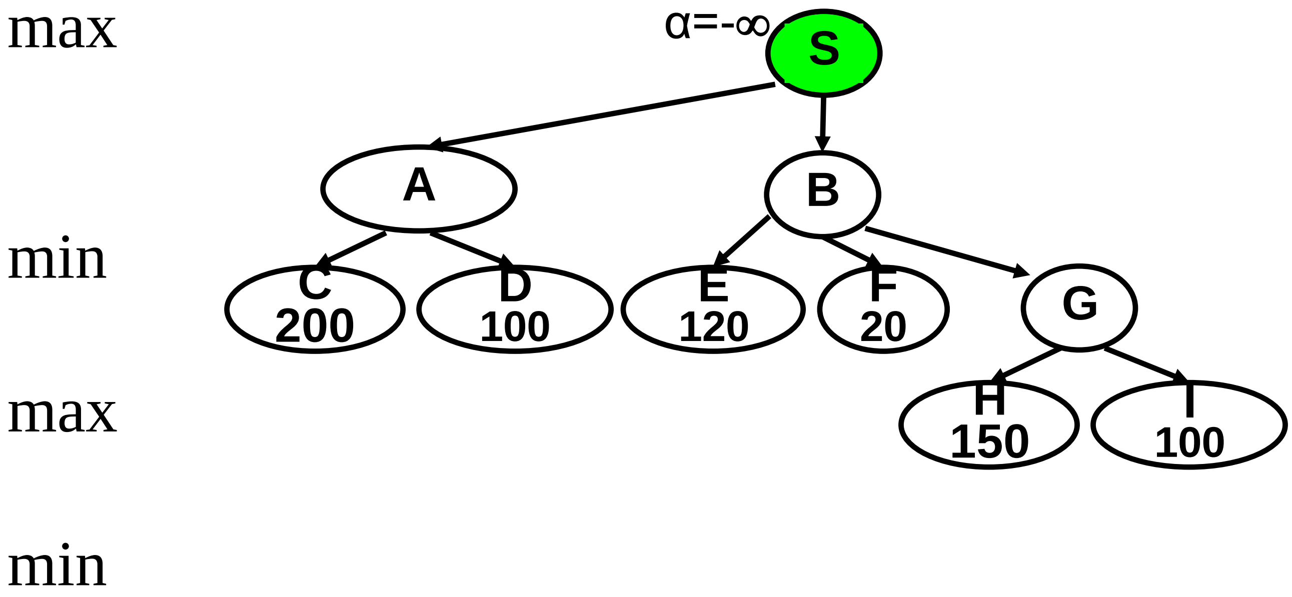
Time complexity?

- $O(b^m)$

Space complexity?

- $O(bm)$

Minimax algorithm in execution



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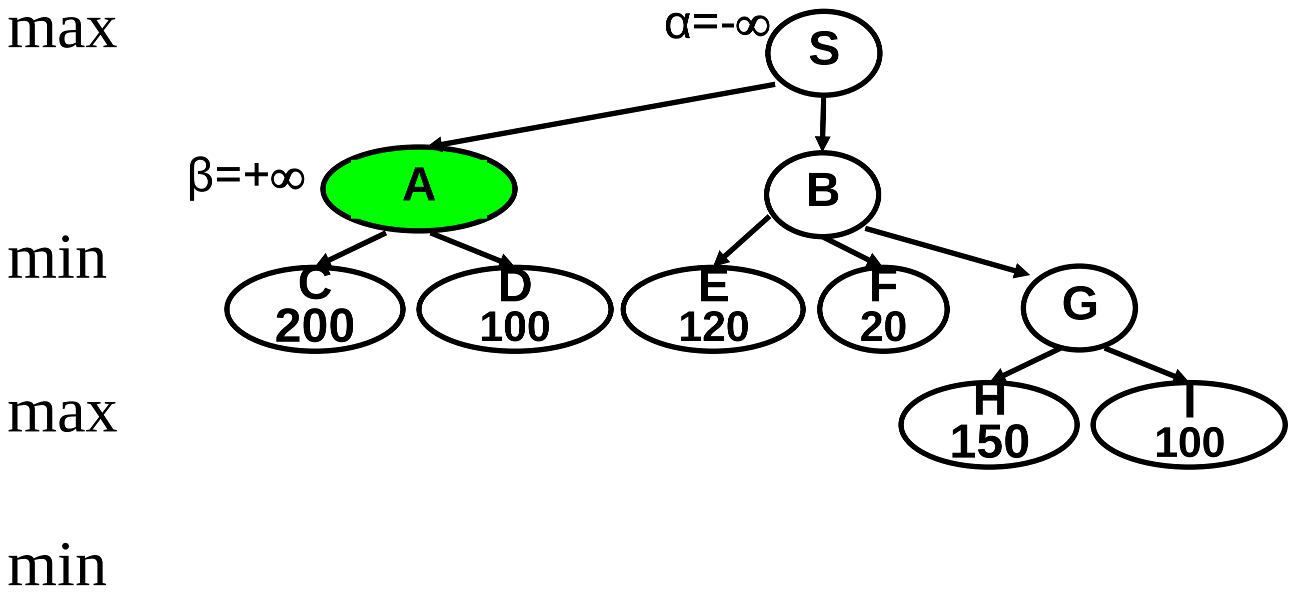

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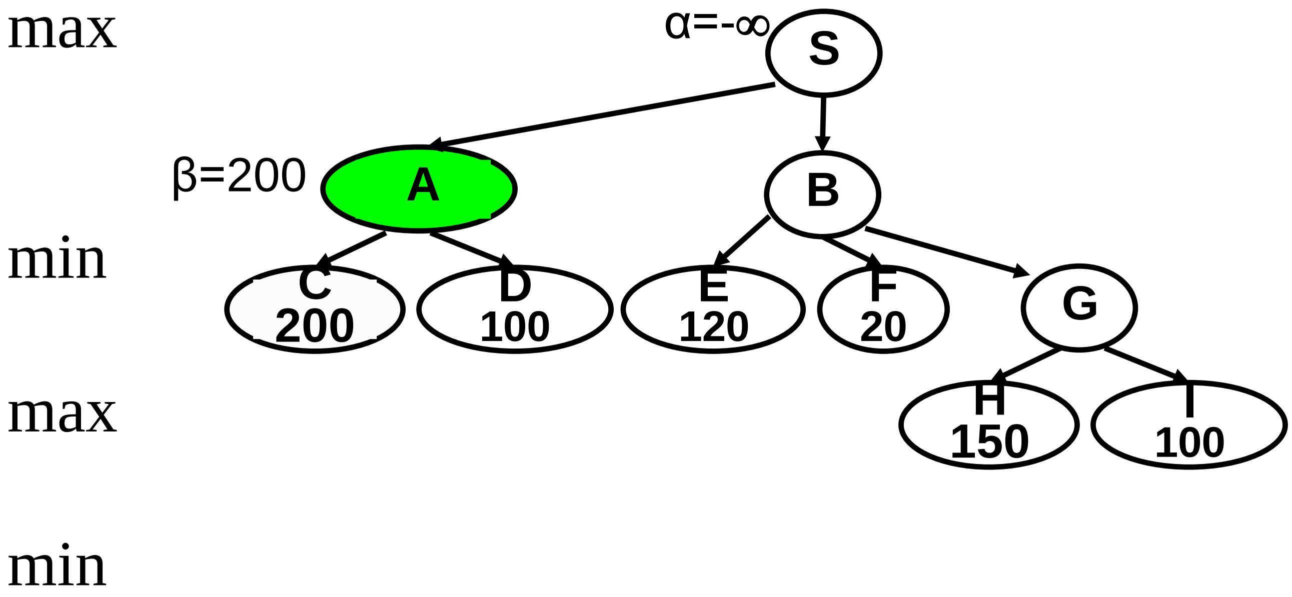
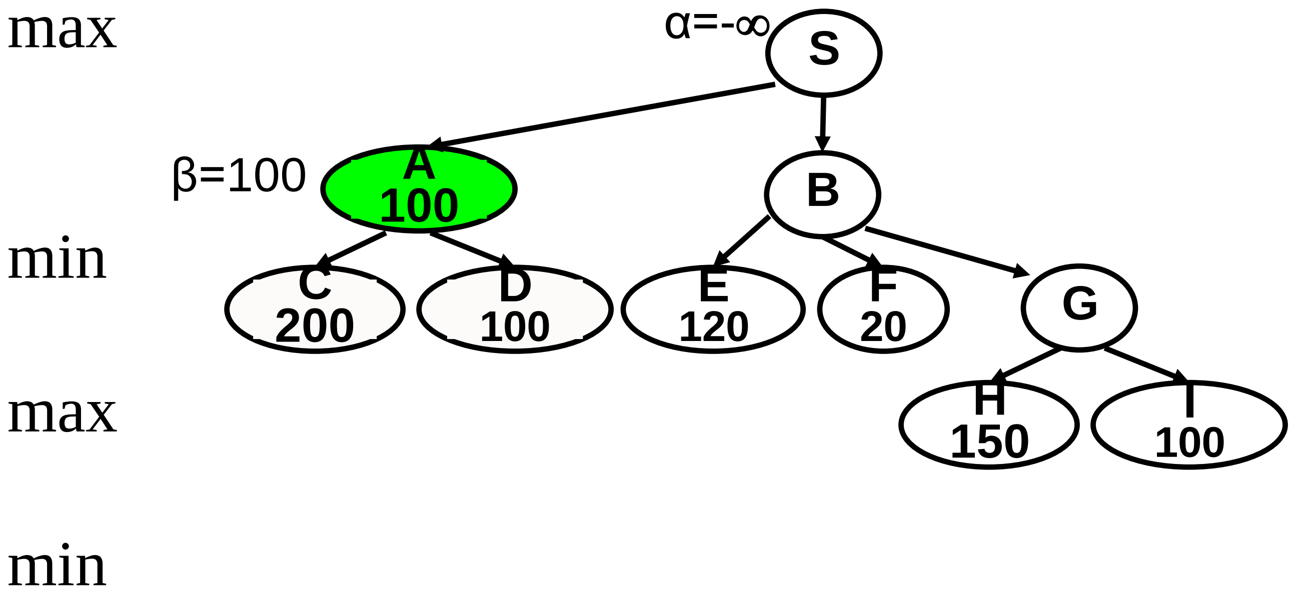


The execution on the terminal nodes is omitted.

Minimax algorithm in execution



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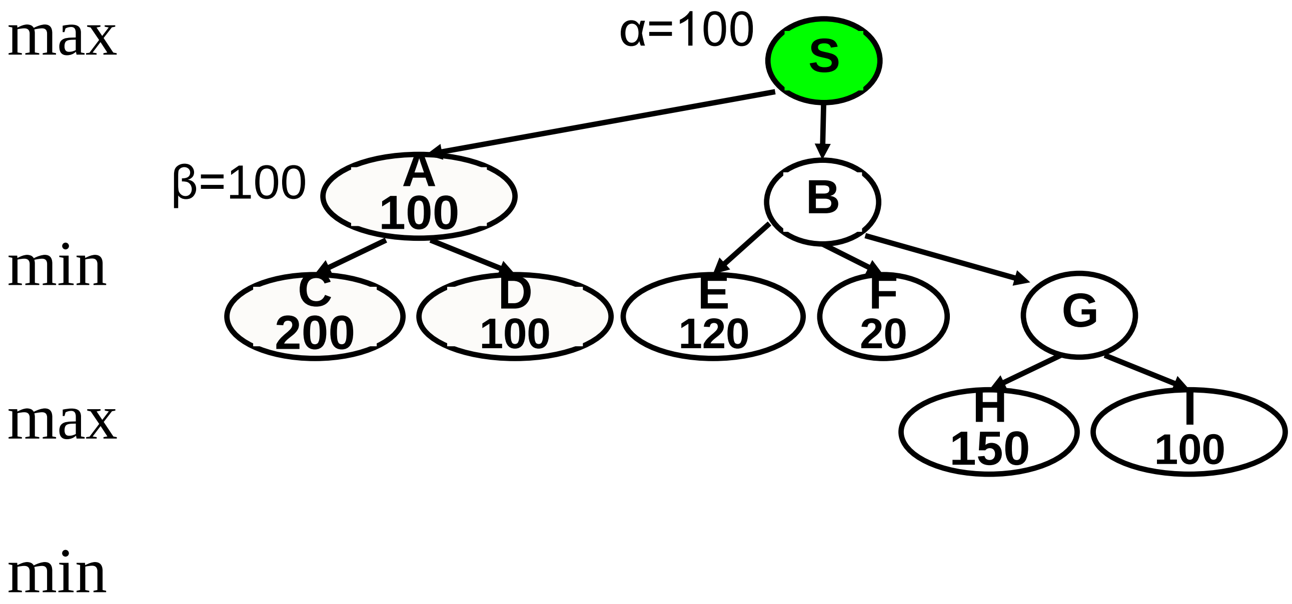

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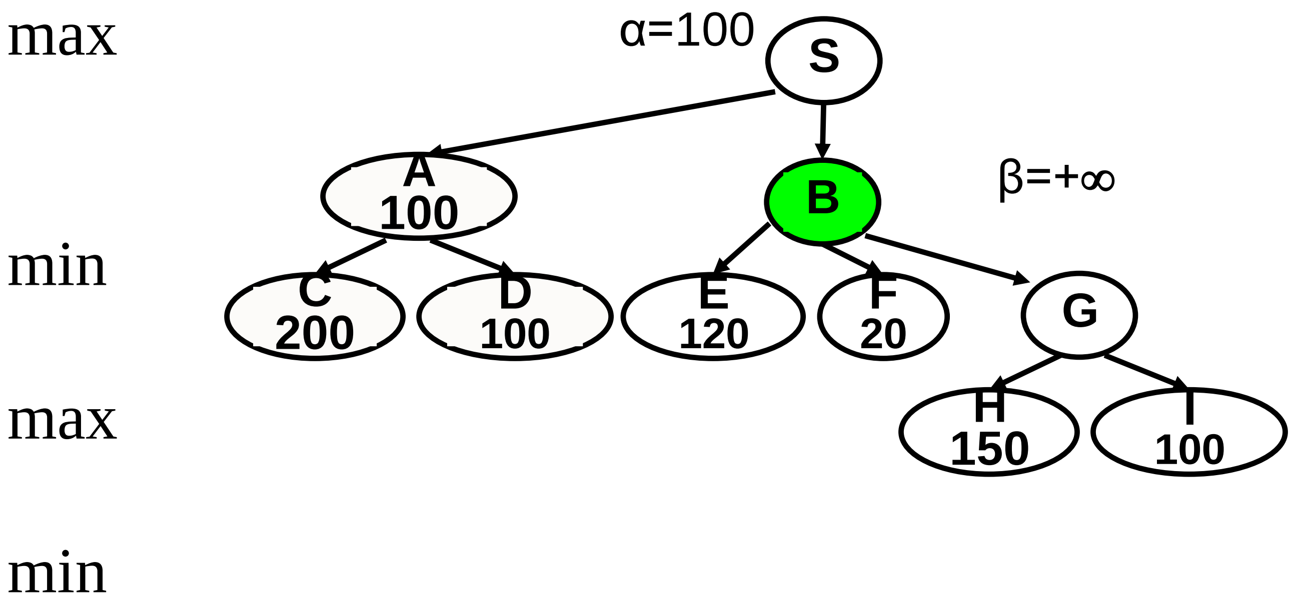

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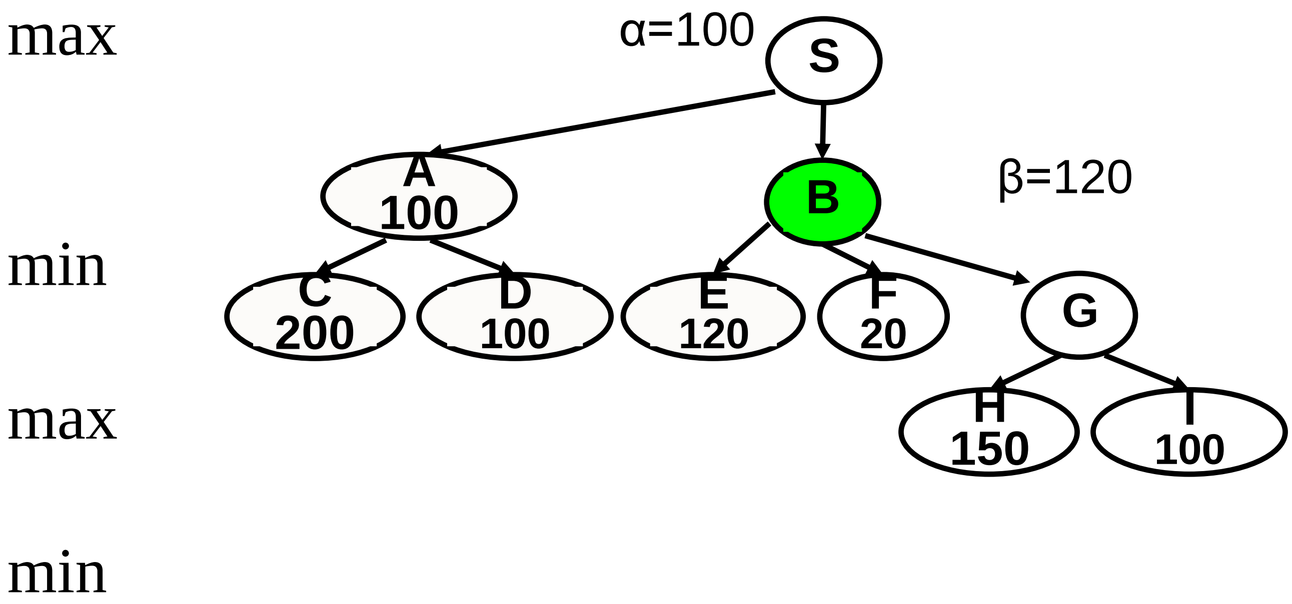

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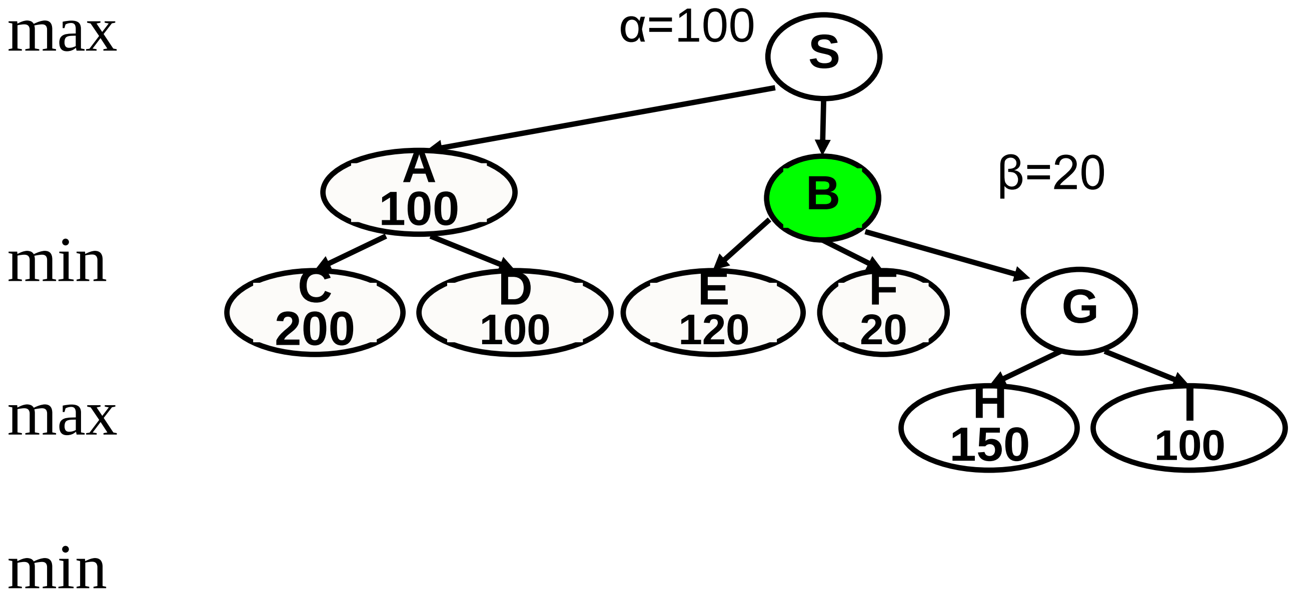

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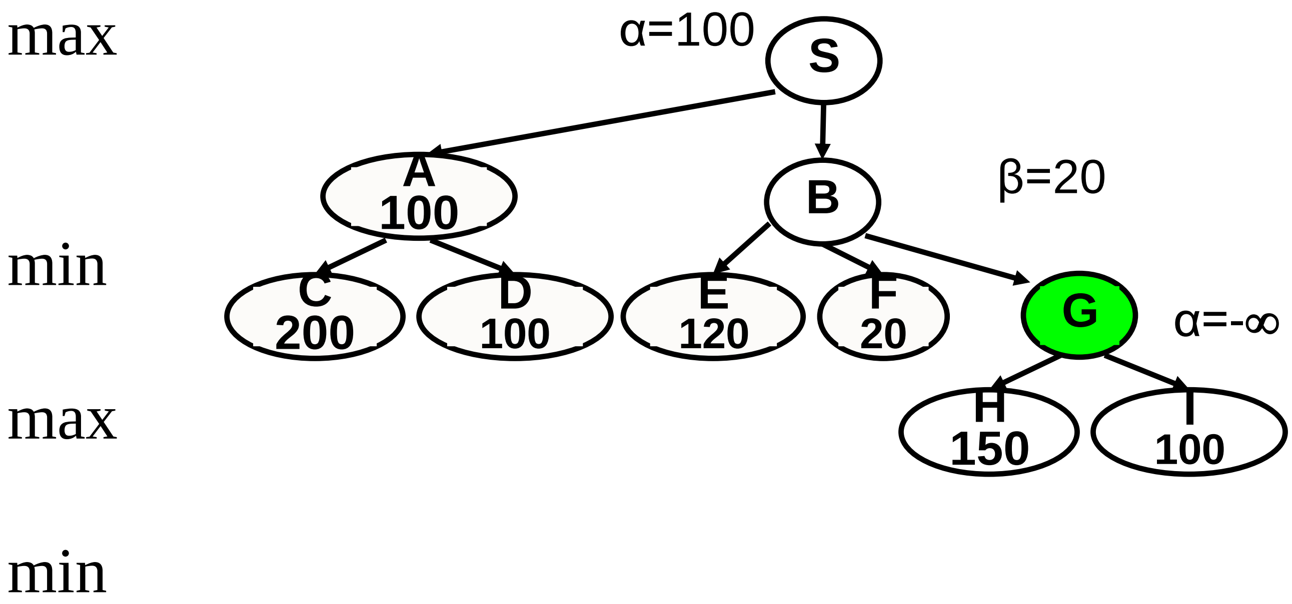

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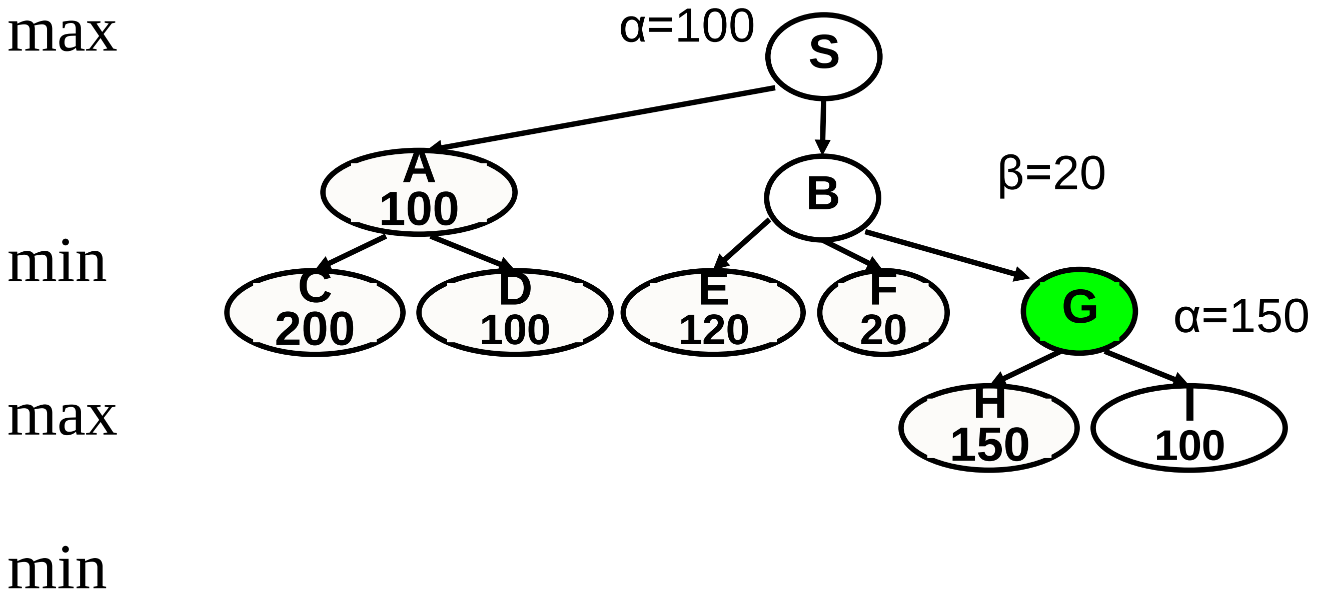

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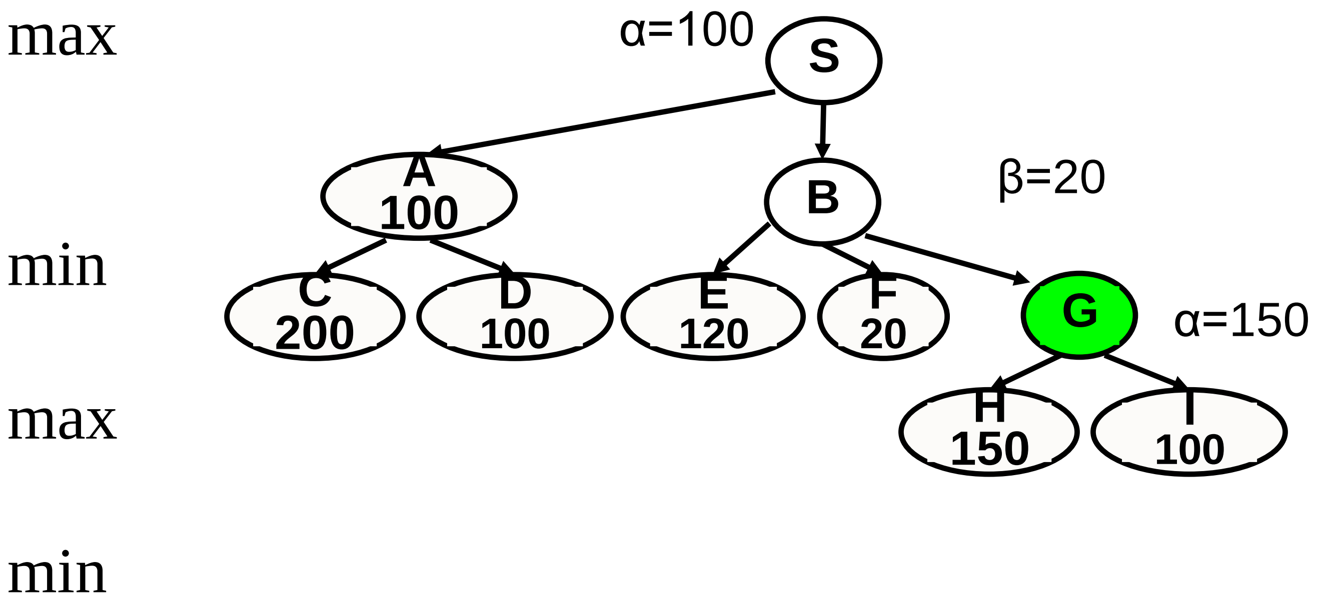

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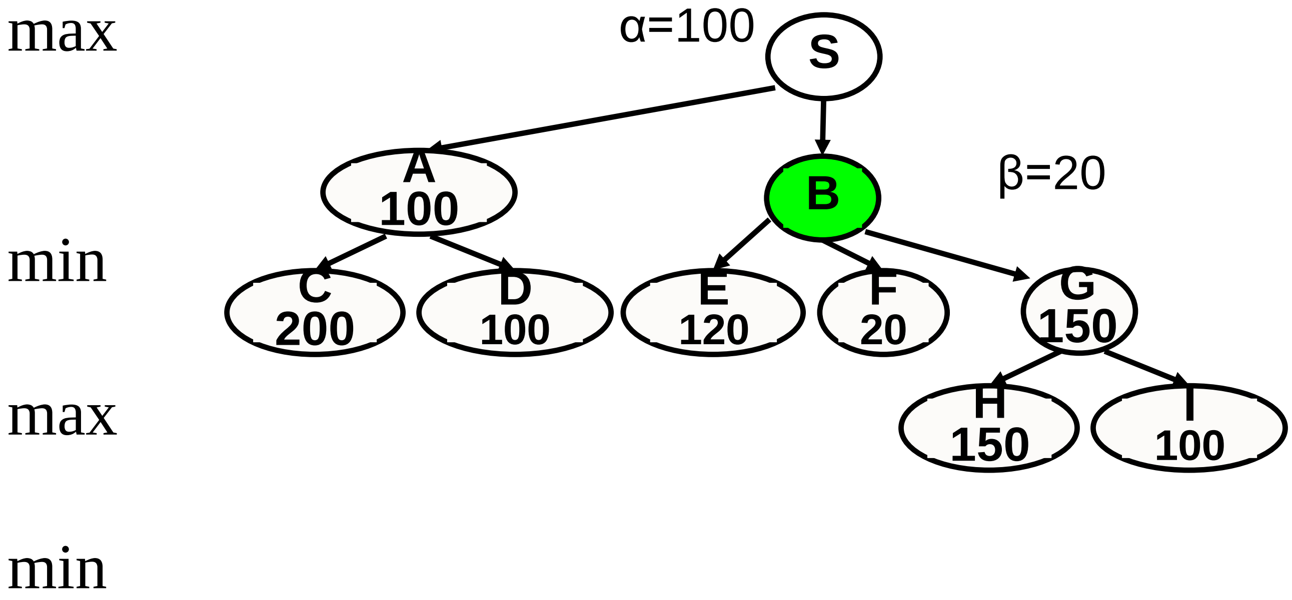

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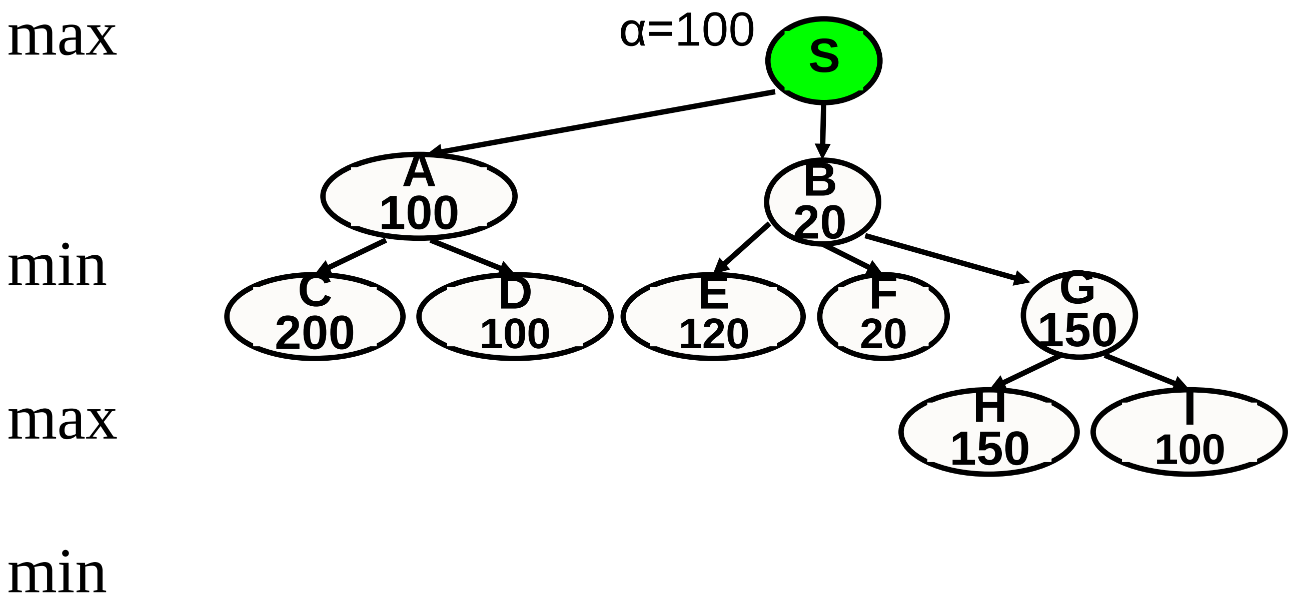
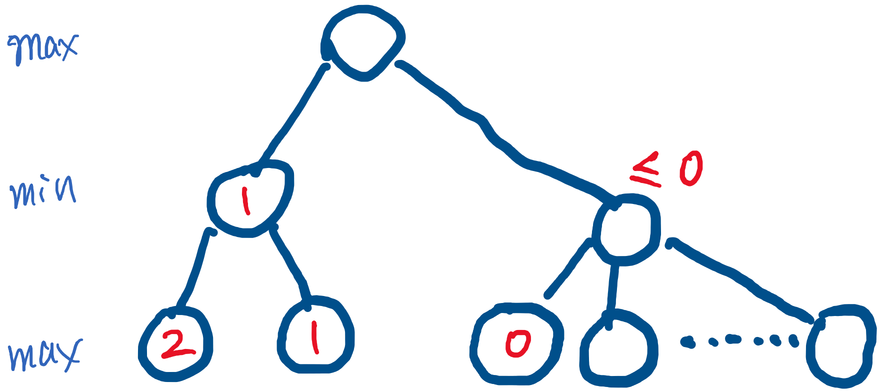


Can We Do Better?

One **downside**: we had to examine the entire tree

An idea to speed things up: **pruning**

- Goal: want the same minimax value, but faster
- We can get rid of bad branches
- Same principle as quiz question



Alpha-beta pruning

```
function Max-Value (s,α,β)
inputs:
      s: current state in game, Max about to play
      α: best score (highest) for Max along path to s
      β: best score (lowest) for Min along path to s
output: min(β , best-score (for Max) available from s)

      if ( s is a terminal state )
      then return ( terminal value of s )
      else for each s’ in Succ(s)
        α := max( α , Min-value(s’,α,β))
        if ( α ≥ β ) then return β   /* alpha pruning */
      return α
```

```
function Min-Value(s,α,β)
output: max(α , best-score (for Min) available from s )

      if ( s is a terminal state )
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        β := min( β , Max-value(s’,α,β))
        if (α ≥ β ) then return α   /* beta pruning */
      return β
```

Starting from the root:

Max-Value(root, -∞, +∞)

Alpha-Beta Pruning

How effective is **alpha-beta pruning**?

- Depends on the order of successors! b^m
  - Best case, the #of nodes to search is $O(b^{m/2}) = O((\sqrt{b})^m)$
  - Happens when each player's best move is the leftmost child.
  - The worst case is no pruning at all.

- In DeepBlue, the average branching factor was about 6 with alpha-beta instead of 35-40 without.

Minimax With Heuristics

Note that long games are yield huge computation

- To deal with this: limit ***d*** for the search depth
- **Q**: What to do at depth *d*, but no termination yet?
  - **A**: Use a heuristic evaluation function *e(x)*

function MINIMAX(x, d) **returns** an estimate of x's utility value
  inputs: x, current state in game
    d, an upper bound on the search depth
  if x is a terminal state **then return** Max's payoff at x
  else if $d = 0$ **then return** $e(x)$
  else if it is Max's move at x **then**
    return $\max\{\text{MINIMAX}(y, d{-}1) : y \text{ is a child of } x\}$
  else return $\min\{\text{MINIMAX}(y, d{-}1) : y \text{ is a child of } x\}$

Credit: Dana Nau

Heuristic Evaluation Functions

- e(x) often a weighted sum of features (like our linear models)

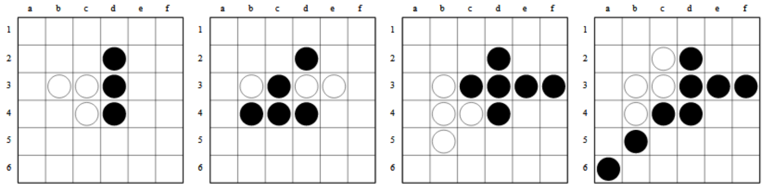
$$e(x) = w_1 f_1(x) + w_2 f_2(x) + \ldots + w_n f_n(x)$$

- Chess example: $f_i(x)$ = **difference** between number of white and black, with *i* ranging over piece types.
  - Set weights according to piece importance
  - E.g., 1(# white pawns - # black pawns) + 3(#white knights - # black knights)

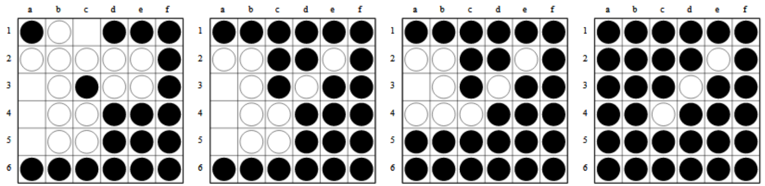
Going Further

- Monte Carlo tree search (MCTS)
  - Uses random sampling of the search space
  - Choose some children (heuristics to figure out #)
  - Record results, use for future play
  - Self-play
- AlphaGo and other big results!

= all we've talk about (pruning / heuristic) + RL + NN



The agent (Black) learns to capture walls and corners in the early game



The agent (Black) learns to force passes in the late game

Credit: Surag Nair

Summary

- Review of game theory
  - Properties, Mathematical formulation for simultaneous games Normal form, dominance, equilibria, mixed vs pure
- Sequential games
  - Game trees, minimax value, minimax algorithm
- Improving our search
  - Using heuristics, pruning, random search

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