Lecture 9–10: Accelerated Gradient Descent

Yudong Chen

In previous lectures, we showed that gradient descent achieves a $\frac{1}{k}$ convergence rate for smooth convex functions and a $(1-\frac{m}{L})^k$ geometric rate for L-smooth and m-strongly convex functions. Gradient descent is very greedy: it only uses the gradient $\nabla f(x_k)$ at the current point to choose the next iterate and discards information from past iterates.

It turns out we can do better than gradient descent, achieving a $\frac{1}{k^2}$ rate and a $\left(1-\sqrt{\frac{m}{L}}\right)^k$ rate in the two cases above. Both rates are optimal in a precise sense. The algorithms the attain these rates are known as *Nesterov's accelerated gradient descent (AGD)* or *Nesterov's optimal methods*.

1 Warm-up: the heavy-ball method

The high level idea of acceleration is adding momentum to the GD update. For example, consider the update

$$
\begin{aligned}
y_k &= x_k + \beta\left(x_k - x_{k-1}\right), && \text{momentum step}\\
x_{k+1} &= y_k - \alpha \nabla f(x_k), && \text{gradient step}
\end{aligned}
$$

where we first take a step in the direction $(x_k - x_{k-1})$, which is the momentum carried over from the previous update, and then take a standard gradient descent step. This is known as Polyak's *heavy-ball method*. The update above is equivalent to a discretization of the second order ODE

$$\ddot{x} = -a\nabla f(x) - b\dot{x},$$

which models the motion of a body in a potential field given by f with friction (hence the name heavy-ball).

It can be shown that for a strongly convex *quadratic* function f, the heavy-ball method achieves the accelerated rate $\left(1-\sqrt{\frac{m}{L}}\right)^k$.[1] For non-quadratic functions (e.g., those that are not twice differentiable), theoretical guarantees for heavy-ball method are less clear; in fact, heavy-ball may not even converge for such functions.

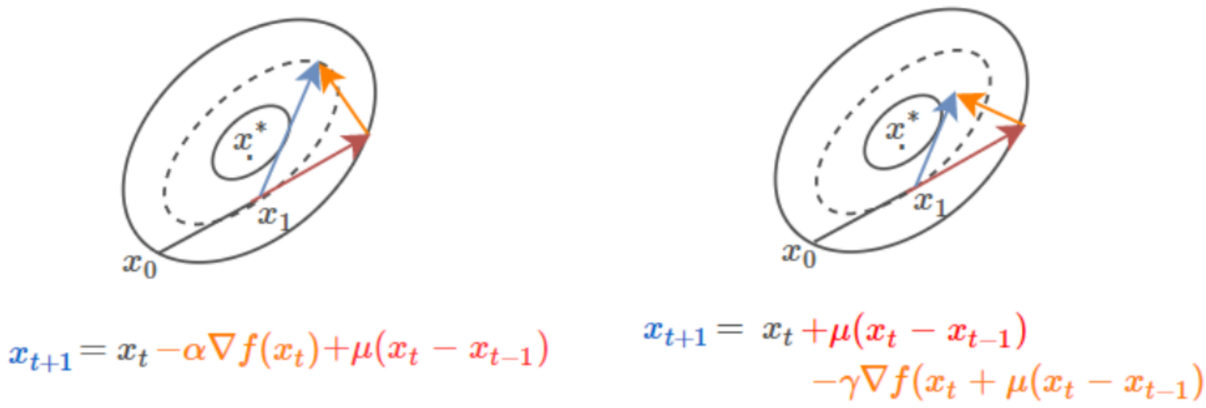
Rather than using the gradient at x_k, Nesterov's AGD uses the gradient at the point y_k *after* the momentum update:

$$
\begin{aligned}
y_k &= x_k + \beta\left(x_k - x_{k-1}\right), && \text{momentum step}\\
x_{k+1} &= y_k - \alpha \nabla f(y_k). && \text{"lookahead" gradient step}
\end{aligned}
$$

As we see below, Nesterov's AGD enjoys convergence guarantees for (strongly) convex functions beyond quadratics.

[1] This rate can be proved using elementary eigenvalue analysis similar to that in Wright-Recht Chap 4.2.

Below is an illustration of the updates of heavy ball method and Nesterov's AGD:[2]



2 AGD for smooth and strongly convex f

Suppose f is m-strongly convex and L-smooth. Nesterov's AGD for minimizing f is given in Algorithm 1.

Algorithm 1 Nesterov's AGD, smooth and strongly convex

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input: initial x_0, strong convexity and smoothness parameters m, L, number of iterations K
initialize: $x_{-1} = x_0$, $\alpha = \frac{1}{L}$, $\beta = \frac{\sqrt{L/m}-1}{\sqrt{L/m}+1}$.
for $k = 0, 1, \dots K$
  $y_k = x_k + \beta (x_k - x_{k-1})$
  $x_{k+1} = y_k - \alpha \nabla f(y_k)$
return x_K

Let x^* be the unique minimizer of f and set $f^* := f(x^*)$. By translation of coordinate, we may assume $x^* = 0$ without loss of generality (hence $x_k = x_k - x^*$ and $y_k = y_k - x_*$). Define $\kappa := \frac{L}{m}$ (condition number), $\rho^2 := 1 - \frac{1}{\sqrt{\kappa}}$ (contraction factor), $u_k := \frac{1}{L} \nabla f(y^k)$, and

$$V_k := f(x_k) - f^* + \frac{L}{2} \left\| x_k - \rho^2 x_{k-1} \right\|_2^2 .$$

The quantity V_k, viewed a function of (x_k, x_{k-1}), is called a Lyapunov/potential function. We will show $V_{k+1} \le \rho^2 V_k$, hence geometric convergence.

By smoothness and strong convexity:

$$f(z) + \langle \nabla f(z), w - z \rangle + \frac{m}{2} \|w - z\|_2^2 \le f(w) \tag{1}$$

$$\le f(z) + \langle \nabla f(z), w - z \rangle + \frac{L}{2} \|w - z\|_2^2, \qquad \forall w, z \tag{2}$$

[2]Credit: Mitliagkas' notes

It follows that

$$
\begin{aligned}
V_{k+1} &= f(x_{k+1}) - f^* + \frac{L}{2}\left\|x_{k+1} - \rho^2 x_k\right\|_2^2 && \text{by defnition} \\
&\leq f(y_k) - f^* + \langle Lu_k, x_{k+1} - y_k\rangle + \frac{L}{2}\left\|x_{k+1} - y_k\right\| + \frac{L}{2}\left\|x_{k+1} - \rho^2 x_k\right\|_2^2 && \text{upper bound (2)} \\
&\leq f(y_k) - f^* - \frac{L}{2}\left\|u_k\right\|_2^2 + \frac{L}{2}\left\|x_{k+1} - \rho^2 x_k\right\|_2^2 && x_{k+1} - y_k = -u_k \\
&= \rho^2\Big[f(y_k) - f^* + L\left\langle u_k, x_k - y_k\right\rangle\Big] - \rho^2 L\left\langle u_k, x_k - y_k\right\rangle && \text{adding and subtracting terms} \\
&\quad + (1-\rho^2)\Big[f(y_k) - f^* - L\left\langle u_k, y_k\right\rangle\Big] + (1-\rho^2)L\left\langle u_k, y_k\right\rangle \\
&\quad - \frac{L}{2}\left\|u_k\right\|_2^2 + \frac{L}{2}\left\|x_{k+1} - \rho^2 x_k\right\|_2^2.
\end{aligned}
$$

But

$$f(y_k) \leq f(x_k) - L\left\langle u_k, x_k - y_k\right\rangle - \frac{m}{2}\left\|x_k - y_k\right\|_2^2 \qquad \text{lower bound (1) with } w = x_k, z = y_k$$

and

$$f(x^*) \geq f(y_k) - L\left\langle u_k, y_k\right\rangle + \frac{m}{2}\left\|y_k\right\|_2^2 \qquad \text{lower bound (1) with } w = x^* = 0, z = y_k.$$

Combining last three equations gives

$$
\begin{aligned}
V_{k+1} &\leq \rho^2\left[f(x_k) - f^* - \frac{m}{2}\left\|x_k - y_k\right\|_2^2\right] - \rho^2 L\left\langle u_k, x_k - y_k\right\rangle \\
&\quad - (1-\rho^2)\frac{m}{2}\left\|y_k\right\|_2^2 + (1-\rho^2)L\left\langle u_k, y_k\right\rangle \\
&\quad - \frac{L}{2}\left\|u_k\right\|_2^2 + \frac{L}{2}\left\|x_{k+1} - \rho^2 x_k\right\|_2^2 \\
&= \rho^2 \underbrace{\left[f(x_k) - f^* + \frac{L}{2}\left\|x_k - \rho^2 x_{k-1}\right\|_2^2\right]}_{V_k} + R_k,
\end{aligned}
$$

where

$$
\begin{aligned}
R_k := &-\rho^2\frac{m}{2}\left\|x_k - y_k\right\|_2^2 - (1-\rho^2)\frac{m}{2}\left\|y_k\right\|_2^2 \\
&+ L\left\langle u_k, y_k - \rho^2 x_k\right\rangle - \frac{L}{2}\left\|u_k\right\|_2^2 \\
&+ \frac{L}{2}\left\|x_{k+1} - \rho^2 x_k\right\|_2^2 - \frac{\rho^2 L}{2}\left\|x_k - \rho^2 x_{k-1}\right\|_2^2.
\end{aligned}
$$

Claim 1. Under the choice of α, β and ρ above, we have

$$R_k = -\frac{1}{2}L\rho^2\left(\frac{1}{\kappa} + \frac{1}{\sqrt{\kappa}}\right)\left\|x_k - y_k\right\|_2^2 \leq 0.$$

Proof. Substitute the definitions of $\alpha, \beta, \rho, x_{k+1}, y_k$ into the definition of R_k. (Verify it yourself!) □

It follows hat $V_{k+1} \leq \rho^2 V_k, \forall k$, hence

$$
\begin{aligned}
f(x_k) - f^* \leq V_k \leq \rho^{2k} V_0 & \\
= \rho^{2k}\left(f(x_0) - f^* + \frac{L}{2}\left\|x_0 - \rho^2 x_0\right\|_2^2\right) & \qquad x_{-1} = x_0 \\
= \rho^{2k}\left(f(x_0) - f^* + \frac{m}{2}\left\|x_0\right\|_2^2\right) & \qquad (1-\rho^2)^2 = \frac{1}{\kappa} = \frac{m}{L} \\
= \rho^{2k}\left(f(x_0) - f^* + \frac{m}{2}\left\|x_0 - x^*\right\|_2^2\right) & \qquad x^* = 0 \\
\leq \rho^{2k}\left(\frac{L}{2}\left\|x_0 - x^*\right\|^2 + \frac{m}{2}\left\|x_0 - x^*\right\|_2^2\right) & \qquad \text{upper bound (2)}, \nabla f(x^*) = 0 \\
= \left(1 - \sqrt{\frac{m}{L}}\right)^k \cdot \frac{L+m}{2}\left\|x_0 - x^*\right\|_2^2. & \qquad \rho^2 = 1 - \sqrt{\frac{m}{L}}
\end{aligned}
\tag{3}
$$

We have established the following.

Theorem 1. *For Nesterov's AGD Algorithm 1 applied to m-strongly convex L-smooth f, we have*

$$f(x_k) - f^* \leq \left(1 - \sqrt{\frac{m}{L}}\right)^k \cdot \frac{(L+m)\left\|x_0 - x^*\right\|_2^2}{2}, \qquad k = 0, 1, \ldots$$

(Iteration complexity bound) Equivalently, we have $f(x_k) - f^* \leq \epsilon$ after at most

$$O\left(\sqrt{\frac{L}{m}} \log \frac{L\left\|x_0 - x^*\right\|_2^2}{\epsilon}\right) \text{ iterations.}$$

Recall GD, which achieves $f(x_k) - f^* = O\left(\left(1 - \frac{m}{L}\right)^k\right)$ and $k = O\left(\frac{L}{m} \log \frac{1}{\epsilon}\right)$. AGD improves by a factor of $\sqrt{\kappa} = \sqrt{\frac{L}{m}}$, which is significant for ill-conditioned problems with a large κ.

Example 1 (Ill-conditioned problems in statistical learning)**.** In statistical learning, we often need to minimize a function of the form

$$f(x) = g(x) + \frac{m}{2}\|x\|_2^2,$$

where g is a convex function corresponding to the empirical risk/training loss (e.g., the logistic regression loss) of a statistical model with parameter x, and $\frac{m}{2}\|x\|_2^2$ is called a regularizer. Often, g is *not* strongly convex, so the strong convexity of f comes from the regularizer. In many settings, the smoothness parameter of f is $O(1)$, and the regularization parameter is taken to be $m \propto \frac{1}{n}$, where n is the number of data points. The condition number $\kappa = \frac{L}{m} \propto n$ can be large in this case. We explore this setting in HW3.

3 AGD for smooth convex f

Suppose f is L-smooth, with a minimizer x^* and minimum value $f^* = f(x^*)$. Nesterov's AGD for such an f is given in Algorithm 2. Note that we allow the momentum parameter β_k to vary with k, and $\lambda_{k+1} \geq 0$ is chosen to satisfy $\lambda_{k+1}^2 - \lambda_{k+1} = \lambda_k^2$.

Algorithm 2 Nesterov's AGD, smooth convex

input: initial x_0, smoothness parameter L, number of iterations K
initialize: $x_{-1} = x_0$, $\alpha = \frac{1}{L}$, $\lambda_0 = 0$, $\beta_0 = 0$.
for $k = 0, 1, \ldots, K$
  $y_k = x_k + \beta_k (x_k - x_{k-1})$
  $x_{k+1} = y_k - \alpha \nabla f(y_k)$
  $\lambda_{k+1} = \frac{1+\sqrt{1+4\lambda_k^2}}{2}$, $\beta_{k+1} = \frac{\lambda_k - 1}{\lambda_{k+1}}$
return x_K

(The Lyapunov function approach in the previous section can be adapted to analyze Algorithm 2; see Wright-Recht Chapter 4.4. Here we present a somewhat different proof.)

Recall that a gradient step satisfies the descent property (Descent Lemma, Lec 6 Lemma 1)

$$f(x_{k+1}) \leq f(y_k) - \frac{1}{2L} \|\nabla f(y_k)\|_2^2 \leq f(y_k). \tag{4}$$

Therefore, we have

$$\begin{aligned} f(x_{k+1}) - f(x_k) &= f(x_{k+1}) - f(y_k) + f(y_k) - f(x_k) \\ &\leq -\frac{1}{2L} \|\nabla f(y_k)\|_2^2 + \langle \nabla f(y_k), y_k - x_k \rangle && \text{descent property (4), convexity} \\ &= -\frac{L}{2} \|y_k - x_{k+1}\|_2^2 + L \langle y_k - x_{k+1}, y_k - x_k \rangle . && \nabla f(y_k) = L(y_k - x_{k+1}) \end{aligned} \tag{5}$$

Similarly:

$$\begin{aligned} f(x_{k+1}) - f(x^*) &= f(x_{k+1}) - f(y_k) + f(y_k) - f(x^*) \\ &\leq -\frac{1}{2L} \|\nabla f(y_k)\|_2^2 + \langle \nabla f(y_k), y_k - x^* \rangle \\ &= -\frac{L}{2} \|y_k - x_{k+1}\|_2^2 + L \langle y_k - x_{k+1}, y_k - x^* \rangle . \end{aligned} \tag{6}$$

Define the optimality gap $\Delta_k := f(x_k) - f(x^*)$. Taking eq.(5)$\times \lambda_k(\lambda_k - 1)$+eq.(6)$\times \lambda_k$, we get

$$\lambda_k(\lambda_k - 1)(\Delta_{k+1} - \Delta_k) + \lambda_k \Delta_{k+1} \leq L \langle y_k - x_{k+1}, \lambda_k(\lambda_k - 1)(y_k - x_k) + \lambda_k(y_k - x^*) \rangle - \frac{L}{2} \lambda_k^2 \|y_k - x_{k+1}\|_2^2 .$$

Rearranging terms gives the key inequality:

$$\lambda_k^2 \Delta_{k+1} - (\lambda_k^2 - \lambda_k)\Delta_k \leq \frac{L}{2} \cdot \left[2 \langle \lambda_k(y_k - x_{k+1}), \lambda_k y_k - (\lambda_k - 1)x_k - x^* \rangle - \|\lambda_k(y_k - x_{k+1})\|_2^2 \right] . \tag{7}$$

As we show below, the parameters λ_k and β_k are chosen to make the LHS and RHS above telescope.

In particular, substituting $\lambda_k^2 - \lambda_k = \lambda_{k-1}^2$ into LHS of (7) and using the identity $2\langle a, b \rangle - \|a\|_2^2 = \|b\|_2^2 - \|b - a\|_2^2$ on RHS of (7), we obtain

$$\lambda_k^2 \Delta_{k+1} - \lambda_{k-1}^2 \Delta_k \leq \frac{L}{2} \cdot \left[\|\lambda_k y_k - (\lambda_k - 1)x_k - x^*\|_2^2 - \|\lambda_k x_{k+1} - (\lambda_k - 1)x_k - x^*\|_2^2 \right] .$$

For the RHS, by definition and our choice of β_{k+1}, we have

$$\begin{aligned} y_{k+1} = x_{k+1} + \beta_{k+1}(x_{k+1} - x_k) = x_{k+1} + \frac{\lambda_k - 1}{\lambda_{k+1}}(x_{k+1} - x_k) \\ \Longleftrightarrow \lambda_{k+1} y_{k+1} - (\lambda_{k+1} - 1)x_{k+1} = \lambda_k x_{k+1} - (\lambda_k - 1)x_k. \end{aligned}$$

Combining the last two equations give

$$\lambda_k^2 \Delta_{k+1} - \lambda_{k-1}^2 \Delta_k \leq \frac{L}{2} \cdot \left[\|\lambda_k y_k - (\lambda_k - 1)x_k - x^*\|_2^2 - \|\lambda_{k+1} y_{k+1} - (\lambda_{k+1} - 1)x_{k+1} - x^*\|_2^2 \right].$$

We sum the above inequalities over k. Note that both sides telescope and $\lambda_0 = 0, \lambda_1 = 1, \beta_1 = -1, y_1 = x_0$, hence

$$\begin{aligned} \lambda_k^2 \Delta_{k+1} - \lambda_0^2 \Delta_1 \leq \frac{L}{2} \|\lambda_1 y_1 - (\lambda_1 - 1)x_1 - x^*\|_2^2 \\ \Longrightarrow \lambda_k^2 \Delta_{k+1} \leq \frac{L}{2} \|x_0 - x^*\|_2^2. \end{aligned}$$

Finally, note that

$$\lambda_k \geq \frac{1 + \sqrt{4\lambda_{k-1}^2}}{2} = \lambda_{k-1} + \frac{1}{2},$$

which, together with $\lambda_1 = 1$, imply $\lambda_k \geq \frac{k+1}{2}, \forall k$. It follows that

$$f(x_{k+1}) - f(x^*) = \Delta_{k+1} \leq \frac{2L \|x_0 - x^*\|_2^2}{(k+1)^2}.$$

We have established the following.

Theorem 2. *For Nesterov's AGD Algorithm 2 applied to L-smooth convex f, we have*

$$f(x_k) - f(x^*) \leq \frac{2L \|x_0 - x^*\|_2^2}{k^2}, \qquad k = 0, 1, \ldots$$

(Iteration complexity bound) Equivalently, we have $f(x_k) - f^* \leq \epsilon$ after at most

$$O\left(\sqrt{\frac{L \|x_0 - x^*\|_2^2}{\epsilon}}\right) \text{ iterations.}$$

Compare with GD, which achieves $f(x_k) - f^* = O\left(\frac{1}{k}\right)$ and $k = O\left(\frac{L}{\epsilon}\right)$. Significant improvement by AGD.

4 Bibliographical notes (optional)

AGD was originally developed in Nesterov (1983). See Nesterov (2004) for a textbook convergence analysis of AGD using bounding functions.

The last decade has witnessed a surge of papers that provide alternative derivation, interpretation or analysis of AGD:

- The Lyapunov function approach in Section 2 follows Lessard et al (2016). In a related direction, Su, Boyd and Candes (2015) connect AGD to a certain second-order ODE. Also related in spirit is a paper by Flammarion and Bach (2015).
- The proof in Section 3 follows Beck and Teboulle (2009).
- Allen-Zhu and Orrechia (2014) view AGD as a linear coupling of GD and mirror descent.
- This blog post by Hardt (2013) relates AGD to Chebyshev polynomials.
- Bubeck et al (2015) provides a geometric perspective and a short proof.
- Diakonikolas and Orecchia (2019) develops the approximate duality gap technique, which applies to the analysis of AGD.

See d'Aspremont et al 2021 for a recent survey on acceleration methods including AGD and beyond.