Lecture 13: Conjugate Gradient Methods: Implementation and Extensions

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1 Recap

Consider $f(x) = \frac{1}{2}x^\top Ax - b^\top x$, where $A \succ 0$. Minimizing f is equivalent to solving the linear system $Ax = b$.

The conjugate gradient (CG) method is given by

$$x_k = \arg\min_{x \in x_0 + \mathcal{K}_k} f(x), \qquad k = 1, 2, \ldots,$$

where $\mathcal{K}_k := \operatorname{Lin}\left\{A(x_0 - x^*), \ldots, A^k(x_0 - x^*)\right\}$ is the *Krylov subspace* of order k.

Lemma 1. *For any $k \geq 1$, we have $\mathcal{K}_k = \operatorname{Lin}\left\{\nabla f(x_0), \ldots, \nabla f(x_{k-1})\right\}$.*

Lemma 2. *For any $0 \leq i < k$, we have $\langle \nabla f(x_k), \nabla f(x_i) \rangle = 0$.*

Corollary 1. *CG finds $x^* = \arg\min_{x \in \mathbb{R}^d} f(x)$ in at most d iterations.*

Corollary 2. $\forall p \in \mathcal{K}_k$, $\langle \nabla f(x_k), p \rangle = 0$.

2 Efficient implementation of CG

Define $\delta_i := x_{i+1} - x_i$.

Lemma 3. *For all $k \geq 1$, $\mathcal{K}_k = \operatorname{Lin}\left\{\delta_0, \delta_1, \ldots, \delta_{k-1}\right\}$.*

Proof. Suppose $\operatorname{Lin}\left\{\delta_0, \delta_1, \ldots, \delta_{k-1}\right\} = \mathcal{K}_k$. Want to show $\operatorname{Lin}\left\{\delta_0, \delta_1, \ldots, \delta_k\right\} = \mathcal{K}_{k+1}$.

- If $\nabla f(x_k) = 0$: In the proof of Lemma 1 we showed that $\mathcal{K}_{k+1} = \mathcal{K}_k$ and $x_{k+1} = x_k = x^*$. Hence $\operatorname{Lin}\left\{\delta_0, \delta_1, \ldots, \delta_{k-1}, \delta_k\right\} = \operatorname{Lin}\left\{\delta_0, \delta_1, \ldots, \delta_{k-1}, 0\right\} = \mathcal{K}_k = \mathcal{K}_{k+1}$.

- If $\nabla f(x_k) \neq 0$: In the proof of Lemma 1 we showed that

$$x_{k+1} = x_0 + \sum_{i=1}^{k} \beta_{i,k+1} A^i (x_0 - x^*) + \beta_{k+1,k+1} A^i (x_0 - x^*)$$

  for some $\beta_{k+1,k+1} \neq 0$, hence

$$\delta_k = x_{k+1} - x_k = \underbrace{x_0 - x_k}_{\in \mathcal{K}_k} + \underbrace{\sum_{i=1}^{k} \beta_{i,k+1} A^i (x_0 - x^*)}_{\in \mathcal{K}_k} + \beta_{k+1,k+1} A^{k+1} (x_0 - x^*),$$

hence

$$\begin{aligned}\mathrm{Lin}\{\delta_0, \delta_1, \ldots, \delta_{k-1}, \delta_k\} &= \mathrm{Lin}\{\mathcal{K}_k \cup \delta_k\} \\ &= \mathrm{Lin}\left\{\mathcal{K}_k \cup A^{k+1}(x_0 - x^*)\right\} \\ &= \mathcal{K}_{k+1}.\end{aligned}$$

□

Lemma 4 (Lemma 1.3.3 in Nesterov's book)**.** *For any $k, i \geq 0, k \neq i$, the vectors δ_i, δ_k are conjugate w.r.t. A, i.e., $\langle A\delta_k, \delta_i\rangle = 0$.*

Proof. Assume w.l.o.g. $k > i$. Then

$$\begin{aligned}\langle A\delta_k, \delta_i\rangle &= \langle A(x_{k+1} - x_k), \delta_i\rangle \\ &= \langle A(x_{k+1} - x^*) - A(x_k - x^*), \delta_i\rangle \\ &= \langle \nabla f(x_{k+1}), \delta_i\rangle - \langle \nabla f(x_k), \delta_i\rangle \\ &= 0 - 0,\end{aligned}$$

where in the last step we use $\delta_i \in \mathcal{K}_{i+1} \subseteq \mathcal{K}_k \subseteq \mathcal{K}_{k+1}$ and Corollary 2 ($\nabla f(x_{k+1}) \perp \mathcal{K}_{k+1}, \nabla f(x_k) \perp \mathcal{K}_k$). □

We are ready to derive an explicit formula for CG iterate x_{k+1}. As $\mathcal{K}_k = \mathrm{Lin}\{\delta_0, \ldots, \delta_{k-1}\}$, we can express $x_{k+1} \in x_0 + \mathcal{K}_{k+1}$ as

$$x_{k+1} = \underbrace{x_k}_{\in x_0 + \mathcal{K}_k} - \underbrace{h_k \nabla f(x_k)}_{\in \mathcal{K}_{k+1} \setminus \mathcal{K}_k} + \underbrace{\sum_{j=0}^{k-1} \alpha_j \delta_j}_{\in \mathcal{K}_k}$$

for some scalars $h_k, \alpha_0, \alpha_1, \ldots, \alpha_{k-1}$. Equivalently,

$$\delta_k = -h_k \nabla f(x_k) + \sum_{j=0}^{k-1} \alpha_j \delta_j.$$

To make the above implementable, we need to determine h_k and $\{\alpha_j\}$. For $i = 0, 1, \ldots, k-1$, taking the inner product with $A\delta_i$ gives

$$\begin{aligned}0 &= \langle A\delta_i, \delta_k\rangle && \text{Lemma 4} \\ &= -h_k \langle A\delta_i, \nabla f(x_k)\rangle + \sum_{j=0}^{k-1} \alpha_j \langle A\delta_j, \delta_i\rangle \\ &= -h_k \langle A\delta_i, \nabla f(x_k)\rangle + \alpha_i \langle A\delta_i, \delta_i\rangle. && \text{Lemma 4}\end{aligned}$$

But

$$A\delta_i = A(x_{i+1} - x^*) - A(x_i - x^*) = \nabla f(x_{i+1}) - \nabla f(x_i).$$

Combining the last two equations gives

$$h_k \langle \nabla f(x_{i+1}) - \nabla f(x_i), \nabla f(x_k)\rangle = \alpha_i \langle A\delta_i, \delta_i\rangle.$$

- For $i = 0, 1, \dots, k-2$, we have $\langle \nabla f(x_{i+1}), \nabla f(x_k) \rangle = \langle \nabla f(x_i), \nabla f(x_k) \rangle = 0$ by Lemma 2, hence
$$0 = \alpha_i \langle A\delta_i, \delta_i \rangle \quad \stackrel{A \succ 0}{\Longrightarrow} \quad \alpha_i = 0.$$

- For $i = k-1$, we have
$$h_k \langle \nabla f(x_k) - \nabla f(x_{k-1}), \nabla f(x_k) \rangle = \alpha_{k-1} \underbrace{\langle A\delta_{k-1}, \delta_{k-1} \rangle}_{\neq 0 \text{ as } A \succ 0}.$$

  Note that $\langle \nabla f(x_{k-1}), \nabla f(x_k) \rangle = 0$, hence
$$\alpha_{k-1} = \frac{h_k \|\nabla f(x_k)\|_2^2}{\langle A\delta_{k-1}, \delta_{k-1} \rangle} = \frac{h_k \|\nabla f(x_k)\|_2^2}{\langle \nabla f(x_k) - \nabla f(x_{k-1}), \delta_{k-1} \rangle}.$$

Combining, we obtain that
$$\begin{aligned} x_{k+1} &= x_k - h_k \nabla f(x_k) + \alpha_{k-1}\delta_{k-1} \\ &= x_k - h_k \underbrace{\left(\nabla f(x_k) - \frac{\|\nabla f(x_k)\|_2^2}{\langle \nabla f(x_k) - \nabla f(x_{k-1}), \delta_{k-1} \rangle} \delta_{k-1} \right)}_{=: p_k}, \end{aligned} \tag{1}$$

where $p_k \in \mathbb{R}^d$ is viewed as the search direction and $h_k \in \mathbb{R}$ is viewed as the stepsize. Since $x_k - hp_k \in x_0 + \mathcal{K}_{k+1}$ for all h and x_{k+1} minimizes $f(x)$ over $\mathcal{K}_{k+1}$, the stepsize h_k is given by exact line search:
$$h_k = \arg\min_{h \in \mathbb{R}} f(x_k - hp_k).$$

Explicit form of CG: In summary, CG can be implemented as
$$x_{k+1} = x_k - h_k p_k,$$

where
$$\begin{aligned} p_k &= \nabla f(x_k) - \frac{\|\nabla f(x_k)\|_2^2}{\langle \nabla f(x_k) - \nabla f(x_{k-1}), \delta_{k-1} \rangle} \delta_{k-1}, \\ \delta_{k-1} &= x_k - x_{k-1}, \\ h_k &= \arg\min_{h \in \mathbb{R}} f(x_k - hp_k). \end{aligned}$$

Note that the exact line search step involves optimizing a one-dimensional quadratic function and can be computed in closed form.

Question 1. *How much storage is needed in CG? How much computation per iteration?*

Remark 1 (Conjugacy). The search directions $p_k = -\frac{1}{h_k}\delta_k$ are conjugate w.r.t. A:
$$\langle Ap_k, p_i \rangle = 0, \qquad \forall k \neq i$$

since $\langle A\delta_k, \delta_i \rangle = 0$ (Lemma 4).

Remark 2 (Relation to heavy-ball). From (1) we have

$$x_{k+1} = x_k - h_k \nabla f(x_k) + \alpha_{k-1}(x_k - x_{k-1}),$$

which resembles the heavy-ball method (gradient step + momentum step) but with time-varying h_k and α_k.

Remark 3. CG does not require knowing the smoothness and strong convexity parameters L and m.

Remark 4. CG for quadratic f has a very rich convergence theory beyond the asymptotic linear rate. For example:

- If A has r distinct eigenvalues, CG terminates in at most r iterations.
- More generally, CG converges fast when the eigenvalues of A have a clustering structure.
- Precondition CG: one may transform the problem so that A has a more favorable eigenvalue distribution.

We will not delve into these results; see Chapter 5.1 of Nocedal-Wright.

3 Extension to non-quadratic functions

We have written CG in a form that only involves the gradient of f, without explicit dependence on the quadratic structure of f. This allows extension to non-quadratic functions. (Such extensions are known as "nonlinear CG", since $\nabla f(x)$ is nonlinear in x.)

Algorithm 1 Nonlinear CG

- Initial search direction: $p_0 = \nabla f(x_0)$.
- For $k = 0, 1, \ldots$
  - Set
    $$x_{k+1} = x_k - h_k p_k,$$
    where h_k is computed by (exact or inexact) line search.
  - Compute the next search direction as
    $$p_{k+1} = \nabla f(x_{k+1}) - \beta_k p_k,$$
    with some specific choice of β_k (see below).

There are different ways of choosing β_k's:

- Dai-Yuan: $\beta_k = \frac{\|\nabla f(x_{k+1})\|_2^2}{\langle \nabla f(x_{k+1}) - \nabla f(x_k), p_k \rangle}$. (equivalent to the α_{k-1} that we derived for quadratic f)
- Fletcher-Rieves: $\beta_k = -\frac{\|\nabla f(x_{k+1})\|_2^2}{\|\nabla f(x_k)\|_2^2}$.
- Polak-Ribiere: $\beta_k = -\frac{\langle \nabla f(x_{k+1}), \nabla f(x_{k+1}) - \nabla f(x_k) \rangle}{\|\nabla f(x_k)\|_2^2}$.

All of above lead to the same results in the case of quadratic f. See Chapter 5.2 of Nocedal-Wright for more on nonlinear CG.

Nonlinear CG is attractive in practice: it does not require matrix storage and performs well empirically (e.g., faster than GD). Theoretical results are not as strong as AGD—this is a topic for further research.