Lecture 16: Frank-Wolfe (aka Conditional Gradient) Method

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1 Setup

Consider the constrained problem

$$\min_{x \in \mathcal{X}} f(x), \tag{P}$$

We still assume that f is L-smooth and convex, and $\mathcal{X}$ is closed, convex and non-empty.

In many settings, computing projection onto $\mathcal{X}$ is expensive, but linear optimization $\min_{x \in \mathcal{X}} c^\top x$ is easy. This is typical when $\mathcal{X}$ is a polytope $\{x \in \mathbb{R}^d : a_i^\top x \le b_i, i = 1, \ldots, m\}$.

Examples:

- Probability simplex and ℓ_1 ball: Projection uses $\Theta(d \log d)$ arithmetics operations (sorting). Linear optimization oracle only takes $\Theta(d)$ (finding the smallest element of the gradient c). This is not a dramatic difference, but linear optimization has other benefits such as sparsity of solution. See Section 5.
- For some polytopes, projection (exactly) is computationally hard, but LP is poly-time. E.g., matching polytope for a general graph with $|V|$ vertices has $\sim 2^{|V|}$ constraints, but LP is tractable (e.g., using Edmonds' algorithm).

Frank-Wolfe (FW) method uses a linear optimization oracle instead of a projection oracle.

2 Frank-Wolfe method

Algorithm 1 Frank-Wolfe

- Input: initial point $x_0 \in \mathcal{X}$, algorithm parameters $a_k > 0, k = 0, 1, \ldots$
- For $k = 0, 1, \ldots$

$$v_k = \operatorname*{argmin}_{u \in \mathcal{X}} \langle \nabla f(x_k), u \rangle,$$
$$x_{k+1} = \frac{A_{k-1}}{A_k} x_k + \frac{a_k}{A_k} v_k,$$

where $A_k = \sum_{i=0}^k a_i = A_{k-1} + a_k$.

Observe that $v_k \in \mathcal{X}$ by definition, hence

$$x_{k+1} = \left(1 - \frac{a_k}{A_k}\right) x_k + \frac{a_k}{A_k} v_k \in \mathcal{X}, \qquad \forall k$$

by convexity of $\mathcal{X}$ and induction.

3 Convergence rate of Frank-Wolfe

We introduce a new style of analysis.

1. We will maintain an upper bound $U_k \geq f(x_{k+1})$ and a lower bound $L_k \leq f(x^*)$. Consequently, the difference $G_k := U_k - L_k$ is an upper bound on the optimality gap $f(x_{k+1}) - f(x^*)$.

2. Recall that $A_k := \sum_{i=0}^{k} a_i$, which is strictly increasing in k. We will show that
$$A_k G_k \leq A_{k-1} G_{k-1} + E_k,$$
where E_k is some "error" term. This implies that
$$G_k \leq \frac{A_0 G_0 + \sum_{i=1}^{k} E_i}{A_k}.$$

3. We will choose $\{a_k\}$ so that $A_0 G_0 + \sum_{i=1}^{k} E_i$ grows slowly with k compared to A_k, hence G_k converges to 0 quickly.

Let us apply the above strategy to FW.

Upper bound: Simply take $U_k = f(x_{k+1})$. Then
$$A_k U_k - A_{k-1} U_{k-1} = A_k f(x_{k+1}) - A_{k-1} f(x_k).$$

Lower bound: We have
$$\begin{aligned}
f(x^*) &\geq \frac{1}{A_k} \sum_{i=0}^{k} a_i \Big(f(x_i) + \langle \nabla f(x_i), x^* - x_i \rangle \Big) && \begin{array}{c}\text{convexity of } f \\ \text{weighted average of lower bounds is also a lower bound}\end{array} \\
&\geq \frac{1}{A_k} \sum_{i=0}^{k} a_i f(x_i) + \frac{1}{A_k} \sum_{i=0}^{k} a_i \min_{u \in \mathcal{X}} \langle \nabla f(x_i), u - x_i \rangle \\
&= \frac{1}{A_k} \sum_{i=0}^{k} a_i f(x_i) + \frac{1}{A_k} \sum_{i=0}^{k} a_i \langle \nabla f(x_i), v_i - x_i \rangle && \text{definition of } v_i \\
&=: L_k.
\end{aligned}$$

Then
$$A_k L_k - A_{k-1} L_{k-1} = a_k f(x_k) + a_k \langle \nabla f(x_k), v_k - x_k \rangle.$$

Evolution of $A_k G_k$: Define $D := \max_{x,y \in \mathcal{X}} \|x - y\|_2$, which is the diameter of $\mathcal{X}$. Then for $k \geq 1$:
$$\begin{aligned}
& A_k G_k - A_{k-1} G_{k-1} \\
&= (A_k U_k - A_{k-1} U_{k-1}) - (A_k L_k - A_{k-1} L_{k-1}) \\
&= A_k \left(f(x_{k+1}) - f(x_k) \right) - a_k \langle \nabla f(x_k), v_k - x_k \rangle && A_{k-1} + a_k = A_k \\
&\leq \cancel{A_k \langle \nabla f(x_k), x_{k+1} - x_k \rangle} + \frac{A_k L}{2} \|x_{k+1} - x_k\|_2^2 - \cancel{a_k \langle \nabla f(x_k), v_k - x_k \rangle} && \text{smoothness of } f \\
&\overset{\text{(i)}}{=} \frac{a_k^2 L}{2 A_k} \|v_k - x_k\|_2^2 \\
&\leq \frac{a_k^2 L}{2 A_k} D^2, \qquad \longleftarrow \text{this is } E_k && (1)
\end{aligned}$$

where (i) holds because

$$x_{k+1} = \frac{A_{k-1}}{A_k} x_k + \frac{a_k}{A_k} v_k \Longleftrightarrow A_k(x_{k+1} - x_k) = a_k(v_k - x_k) \implies x_{k+1} - x_k = \frac{a_k}{A_k}(v_k - x_k).$$

(Exercise) Using similar argument as above, verify yourself that

$$A_0 G_0 \le \frac{a_0^2 L}{2A_0} D^2. \tag{2}$$

Final bound: Summing (1) over k and (2), we get

$$\begin{aligned} A_k G_k &\le \sum_{i=0}^{k} \frac{a_i^2 L}{2A_i} D^2 \\ \implies f(x_{k+1}) - f(x^*) &\le G_k \le \frac{LD^2}{2} \cdot \frac{1}{A_k} \sum_{i=0}^{k} \frac{a_i^2}{A_i}. \end{aligned}$$

We want to choose $\{a_i\}$ to make RHS to decay fast with k. Different choices work, but whenever you see something like $\frac{a_i^2}{A_i}$, you should try $a_i \propto i \implies A_i \propto i^2, \frac{a_i^2}{A_i} \approx 1$. In particular, setting $a_i = i+1$, we have $A_i = \frac{(i+1)(i+2)}{2}$ and hence

$$f(x_{k+1}) - f(x^*) \le \frac{LD^2}{(k+1)(k+2)} \underbrace{\sum_{i=0}^{k} \frac{2(i+1)^2}{(i+1)(i+2)}}_{\le 2(k+1)} \le \frac{2LD^2}{k+2}.$$

Therefore, we get an $O\left(\frac{LD^2}{k}\right)$ convergence rate. Equivalently, FW achieves $f(x_k) - f(x^*) \le \epsilon$ after at most $O\left(\frac{LD^2}{\epsilon}\right)$ iterations.

4 Lower bound

Is it possible to beat FW? Not in the worst case, if we are only accessing $\mathcal{X}$ via a linear optimization oracle.

Theorem 1. *Consider any algorithm that accesses the feasible set $\mathcal{X}$ only via a linear optimization oracle. There exists an L-smooth convex function function $f : \mathbb{R}^d \to \mathbb{R}$ such that this algorithm requires at least*

$$\min \left\{ \frac{d}{2}, \frac{LD^2}{16\epsilon} \right\}$$

iterations (i.e., calls to the linear optimization oracle) to construct a point $\hat{x} \in \mathcal{X}$ with $f(\hat{x}) - \min_{x \in \mathcal{X}} f(x) \le \epsilon$. The lower bound applies even if f is strongly convex.

Proof sketch. Take $f(x) = \frac{1}{2}\|x\|_2^2$ and $\mathcal{X} = \left\{ x \in \mathbb{R}^d : x \ge 0, \sum_{i=1}^d x_i = 1 \right\}$ (the probability simplex). Note that the smoothness parameter of f is $L = 1$, the diameter of $\mathcal{X}$ is $D = 2$, and f is strongly convex. Moreover, the optimal solution and value are

$$x^* = \frac{1}{d}\mathbf{1} = \frac{1}{d}\sum_{i=1}^{d} e_i, \qquad f(x^*) = \frac{1}{2d},$$

where $e_i = (0, \dots, 0, 1, 0, \dots, 0)^\top$ denotes the i-th standard basis vector.

Linear optimization over the polytope $\mathcal{X}$ returns one of its vertex e_i. After k iterations, one would only uncover k basis vectors $e_{i_1}, e_{i_2}, \dots, e_{i_k}$. The best solution one can construct from them is $\hat{x} = \frac{1}{k}\sum_{j=1}^{k} e_{i_j}$, hence

$$f(\hat{x}) - f(x^*) \geq \frac{1}{2}\left(\frac{1}{\min\{k,d\}} - \frac{1}{d}\right).$$

To make the RHS $\leq \epsilon$, we need $k \geq \min\left\{\frac{d}{2}, \frac{1}{4\epsilon}\right\} = \min\left\{\frac{d}{2}, \frac{LD^2}{16\epsilon}\right\}$.

See Lan '13 for the complete proof. □

5 Additional remarks

FW was out of favor for a long time, as it has sublinear convergence even when f is strongly convex. However, there has been a recent upsurge of activity on FW.

- A sublinear rate is acceptable in many machine learning and data science problems with large-scale and noisy data.
- The optimal solution v_k of linear optimization lies at a vertex of the feasible set $\mathcal{X}$. Such a solution often has certain *sparsity* properties not possessed by projection onto $\mathcal{X}$. Sparsity often leads to better computational and statistical efficiency. For example:
  - When $\mathcal{X}$ is the probability simplex or ℓ_1 ball, each v_i is 1-sparse (has only 1 nonzero entry). Consequently, the iterate x_k of FW is k-sparse since it is a convex combination of $\{v_1, \dots, v_k\}$.
  - The nuclear norm $\|x\|_{\text{nuc}}$ of a matrix x is defined as the sum of its singular values. When $\mathcal{X} = \left\{x \in \mathbb{R}^{d\times d} : \|x\|_{\text{nuc}} \leq R\right\}$ is the nuclear norm ball, each v_i is a rank-1 matrix, hence x_k has rank at most k.
- Conservative Policy Iteration (CPI), a basic algorithm in Reinforcement Learning, is an incarnation of FW. See this short paper on the connection between several reinforcement learning and constrained optimization algorithms (including CPI and FW).