Lecture 15: Projected Gradient Descent

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Consider the problem

$$\min_{x\in\mathcal{X}} f(x), \tag{P}$$

where f is continuously differentiable and $\mathcal{X} \subseteq \mathrm{dom}(f) \subseteq \mathbb{R}^n$ is a closed, convex, nonempty set.

In this lecture, we further assume f is L-smooth (w.r.t. $\|\cdot\|_2$).

1 Projected gradient descent and gradient mapping

Recall the first-order condition for L-smoothness:

$$\forall x, y: \quad f(y) \le f(x) + \langle \nabla f(x), y - x\rangle + \frac{L}{2}\|y-x\|_2^2. \tag{1}$$

For unconstrained problem, recall that each iteration of gradient descent (GD) minimizes the RHS above:

$$\textbf{(GD)} \qquad \begin{aligned} x_{k+1} &= \operatorname*{argmin}_{y\in\mathbb{R}^d} \left\{ f(x_k) + \langle \nabla f(x_k), y - x_k\rangle + \frac{L}{2}\|y-x_k\|_2^2 \right\} \\ &= x_k - \frac{1}{L}\nabla f(x_k). \end{aligned}$$

Projected Gradient Descent (PGD) For constrained problem, we consider PGD, which minimizes the RHS of (1) *over the feasible set* $\mathcal{X}$*:*

$$\textbf{(PGD)} \qquad \begin{aligned} x_{k+1} &= \operatorname*{argmin}_{y\in\mathcal{X}} \left\{ f(x_k) + \underbrace{\langle \nabla f(x_k), y - x_k\rangle + \frac{L}{2}\|y-x_k\|_2^2}_{\text{complete this square}} \right\} \\ &= \operatorname*{argmin}_{y\in\mathcal{X}} \left\{ \frac{L}{2}\left\| y - x_k + \frac{1}{L}\nabla f(x_k)\right\|_2^2 \right\} \\ &= P_{\mathcal{X}}\left(x_k - \frac{1}{L}\nabla f(x_k)\right). \end{aligned}$$

As in GD, we can also use some other stepsize $\frac{1}{\eta}$ with $\eta \ge L$:

$$x_{k+1} = P_{\mathcal{X}}\left(x_k - \frac{1}{\eta}\nabla f(x_k)\right).$$

It will be useful later to recall that Euclidean projection is characterized by the minimum principle

$$\forall y \in \mathcal{X}: \quad \langle P_{\mathcal{X}}(x) - x, y - P_{\mathcal{X}}(x)\rangle \ge 0. \tag{2}$$

1.1 Gradient mapping

Many results for GD can be generalized to PGD, where the role of the gradient is replaced by the gradient mapping defined below.

Definition 1 (Gradient Mapping)**.** Suppose $\mathcal{X} \subseteq \mathbb{R}^d$ is closed, convex and nonempty, and f is differentiable. Given $\eta > 0$, the *gradient mapping* $G_\eta : \mathbb{R}^d \to \mathbb{R}^d$ is defined by

$$G_\eta(x) = \eta \left(x - P_{\mathcal{X}} \left(x - \frac{1}{\eta} \nabla f(x) \right) \right) \qquad \text{for } x \in \mathbb{R}^d.$$

Using the above definition, we can write PGD in a form that resembles GD:

$$x_{k+1} = x_k - \frac{1}{\eta} G_\eta(x_k).$$

The fixed points of PGD are those that satisfy $G_\eta(x) = 0$.

Remark 1. When $\mathcal{X} = \mathbb{R}^d$, $G_\eta(x) = \nabla f(x)$. Hence the gradient mapping generalizes the gradient.

For constrained problems, gradient mapping acts as a "proxy" for the gradient and has properties similar to the gradient.

- If $G_\eta(x) = 0$, then x is a stationary point in the sense that $-\nabla f(x) \in N_{\mathcal{X}}(x)$. If $\left\| G_\eta(x) \right\|_2 \leq \epsilon$, we get a near-stationary point.
- A Descent Lemma holds for PGD: if we use $\eta \geq L$, then $f(x_{k+1}) - f(x_k) \leq -\frac{1}{2\eta} \left\| G_\eta(x_k) \right\|_2^2$.

We elaborate below.

1.2 Gradient mapping and stationarity

The first lemma shows that x^* is a stationary point of (P) if and only if $G_\eta(x^*) = 0$.

Lemma 1 (Wright-Recht Prop 7.8)**.** *Consider (P), where f is L-smooth, and $\mathcal{X}$ is closed, convex and nonempty. Then, $x^* \in \mathcal{X}$ satisfies the first-order condition $-\nabla f(x^*) \in N_{\mathcal{X}}(x^*)$ if and only if $x^* = P_{\mathcal{X}} \left(x^* - \frac{1}{\eta} \nabla f(x^*) \right)$ (equivalently, $G_\eta(x^*) = 0$).*

Proof. **"if" part:** Suppose $G_\eta(x^*) = 0$. This means

$$x^* = P_{\mathcal{X}} \left(x^* - \frac{1}{\eta} \nabla f(x^*) \right) = \operatorname*{argmin}_{y \in \mathcal{X}} \left\{ \frac{1}{2} \left\| y - \left(x^* - \frac{1}{\eta} \nabla f(x^*) \right) \right\|_2^2 \right\}.$$

By first-order optimality condition applied to the above minimization problem, we have

$$N_{\mathcal{X}}(x^*) \ni -\nabla \left[\frac{1}{2} \left\| y - \left(x^* - \frac{1}{\eta} \nabla f(x^*) \right) \right\|_2^2 \right] \Bigg|_{y = x^*} = -\frac{1}{\eta} \nabla f(x^*),$$

which is equivalent to $N_{\mathcal{X}}(x^*) \ni -\frac{1}{\eta} \nabla f(x^*)$.

"only if" part: Suppose $-\nabla f(x^*) \in N_{\mathcal{X}}(x^*)$. By definition of $N_{\mathcal{X}}(x^*)$, we have

$$\begin{aligned}\forall y \in \mathcal{X}: \qquad 0 &\geq \frac{1}{\eta}\langle -\nabla f(x^*), y - x^*\rangle \\ &= \left\langle x^* - \frac{1}{\eta}\nabla f(x^*) - x^*, y - x^*\right\rangle.\end{aligned}$$

By the minimum principle (2) with $x = x^* - \frac{1}{\eta}\nabla f(x^*)$, the above inequality implies

$$x^* = P_{\mathcal{X}}(x) = P_{\mathcal{X}}\left(x^* - \frac{1}{\eta}\nabla f(x^*)\right).$$

□

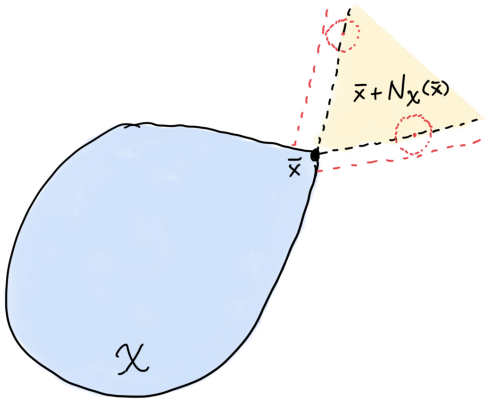
To state the next lemma, we need some notations. Let $\mathcal{B}_2(z,r) := \{x \in \mathbb{R}^d : \|x - z\|_2 \leq r\}$ denotes the Euclidean ball of radius r centered at z. For two sets $S_1, S_2 \subset \mathbb{R}^d$, let $S_1 + S_2 = \{x + y : x \in S_1, y \in S_2\}$ denote their Minkowski sum.

Our next Lemma 2 says if $\|G_\eta(x)\|_2$ is small, then x almost satisfies the first-order optimality condition and can be considered a near-stationary point. Lemma 2 is a generalization of the "if" part of Lemma 1.

Recall that the Minkowski sum of two sets $A, B \subseteq \mathbb{R}^d$ is defined as $A + B := \{a + b : a \in A, b \in B\}$.

Lemma 2 (Gradient mapping as a surrogate for stationarity)**.** *Consider (P), where f is L-smooth, and $\mathcal{X}$ is closed, convex and nonempty. Denote $\bar{x} = P_{\mathcal{X}}\left(x - \frac{1}{\eta}\nabla f(x)\right)$, so that $G_\eta(x) = \eta(x - \bar{x})$. If $\|G_\eta(x)\|_2 \leq \epsilon$ for some $\epsilon \geq 0$, then:*

$$\begin{aligned}&-\nabla f(\bar{x}) \in N_{\mathcal{X}}(\bar{x}) + \mathcal{B}_2\left(0, \epsilon\left(\frac{L}{\eta} + 1\right)\right) \\ \Longleftrightarrow &\forall u \in \mathcal{X}: \langle -\nabla f(\bar{x}), u - \bar{x}\rangle \leq \epsilon\left(\frac{L}{\eta} + 1\right)\|u - \bar{x}\|_2 \\ \Longrightarrow &\forall u \in \mathcal{X} \cap \mathcal{B}_2(\bar{x}, 1): \langle -\nabla f(\bar{x}), u - \bar{x}\rangle \leq \epsilon\left(\frac{L}{\eta} + 1\right).\end{aligned}$$



Proof. Suppose that $\|G_\eta(x)\|_2 \leq \epsilon$. By definition:

$$\bar{x} = P_{\mathcal{X}}\left(x - \frac{1}{\eta}\nabla f(x)\right) = \operatorname*{argmin}_{y \in \mathcal{X}}\left\{\frac{1}{2}\left\|y - \left(x - \frac{1}{\eta}\nabla f(x)\right)\right\|_2^2\right\}.$$

Hence $\bar{x}$ satisfies the optimality condition of the minimization problem above:

$$-\left(\bar{x}-x+\frac{1}{\eta}\nabla f(x)\right)\in N_{\mathcal{X}}(\bar{x}).$$

Adding and subtracting $-\frac{1}{\eta}\nabla f(\bar{x})$:

$$-\frac{1}{\eta}\nabla f(\bar{x})-\underbrace{\left(\bar{x}-x+\frac{1}{\eta}\nabla f(x)-\frac{1}{\eta}\nabla f(\bar{x})\right)}_{\rho}\in N_{\mathcal{X}}(\bar{x}).$$

Note that

$$\begin{aligned}\|\rho\|_2 &= \Big\| \underbrace{\bar{x}-x}_{-\frac{1}{\eta}G_\eta(x)} +\frac{1}{\eta}\left(\nabla f(x)-\nabla f(\bar{x})\right)\Big\|_2\\ &\le \frac{1}{\eta}\left\|G_\eta(x)\right\|_2+\frac{1}{\eta}\underbrace{\|\nabla f(x)-\nabla f(\bar{x})\|_2}_{\le L\|x-\bar{x}\|_2=\frac{L}{\eta}\left\|G_\eta(x)\right\|_2}\\ &\le \frac{1}{\eta}\left(1+\frac{L}{\eta}\right)\left\|G_\eta(x)\right\|_2\\ &\le \frac{\epsilon}{\eta}\left(1+\frac{L}{\eta}\right).\end{aligned}$$

Hence

$$\begin{aligned}&-\frac{1}{\eta}\nabla f(\bar{x})\in N_{\mathcal{X}}(\bar{x})+\rho\\ \iff &-\nabla f(\bar{x})\in N_{\mathcal{X}}(\bar{x})+\eta\rho\\ \implies &-\nabla f(\bar{x})\in N_{\mathcal{X}}(\bar{x})+\mathcal{B}_2\left(0,\epsilon\left(1+\frac{L}{\eta}\right)\right).\end{aligned}$$

□

1.3 Sufficient descent property/descent lemma

The gradient mapping also inherits the descent lemma.

Lemma 3 (Theorem 2.2.13 in Nesterov's 2018 textbook)**.** *Consider (P), where f is an L-smooth function. If $\eta \ge L$, $x\in\mathcal{X}$ and $\bar{x}=x-\frac{1}{\eta}G_\eta(x)$, then:*

$$f(\bar{x})\le f(x)-\frac{1}{2\eta}\left\|G_\eta(x)\right\|_2^2.$$

Proof. From the first-order condition for L-smoothness (Lecture 4, Lemma 1),

$$\begin{aligned}f(\bar{x}) &\le f(x)+\langle\nabla f(x),\bar{x}-x\rangle+\frac{\eta}{2}\left\|\bar{x}-x\right\|_2^2\\ &= f(x)-\frac{1}{\eta}\left\langle\nabla f(x),G_\eta(x)\right\rangle+\frac{1}{2\eta}\left\|G_\eta(x)\right\|_2^2 && \bar{x}-x=-\frac{1}{\eta}G_\eta(x)\\ &= f(x)-\frac{1}{2\eta}\left\|G_\eta(x)\right\|_2^2+\frac{1}{\eta}\left\langle G_\eta(x)-\nabla f(x),G_\eta(x)\right\rangle. && \text{add/subtract } \frac{1}{\eta}\left\langle G_\eta(x),G_\eta(x)\right\rangle=\frac{1}{\eta}\left\|G_\eta(x)\right\|_2^2\end{aligned}$$

It remains to show that $\langle G_\eta(x) - \nabla f(x), G_\eta(x)\rangle \le 0$. Plugging in the definition of $G_\eta(x)$, we have

$$\begin{aligned}
&\langle G_\eta(x) - \nabla f(x), G_\eta(x)\rangle \\
=&\left\langle \eta\left[x - P_{\mathcal{X}}\left(x - \frac{1}{\eta}\nabla f(x)\right)\right] - \nabla f(x), \eta\left[x - P_{\mathcal{X}}\left(x - \frac{1}{\eta}\nabla f(x)\right)\right]\right\rangle \\
=&\eta^2\left\langle \underbrace{x - \frac{1}{\eta}\nabla f(x)}_{=:z} - P_{\mathcal{X}}\left(x - \frac{1}{\eta}\nabla f(x)\right), x - P_{\mathcal{X}}\left(x - \frac{1}{\eta}\nabla f(x)\right)\right\rangle \\
=&\eta^2\langle z - P_{\mathcal{X}}(z), x - P_{\mathcal{X}}(z)\rangle \\
\le&0
\end{aligned}$$

using $x \in \mathcal{X}$ and the minimum principle (2). □

2 Convergence guarantees for projected gradient descent

Consider the PGD update

$$x_{k+1} = P_{\mathcal{X}}\left(x_k - \frac{1}{L}\nabla f(x_k)\right) = x_k - \frac{1}{L}G_L(x_k),$$

where we fix the stepsize to be $\frac{1}{L}$, with L being the smoothness parameter of f.

The convergence guarantees of PGD parallel those of GD.

2.1 Nonconvex case

Suppose f is L-smooth.

By the Descent Lemma 3:

$$f(x_{k+1}) - f(x_k) \le -\frac{1}{2L}\|G_L(x_k)\|_2^2.$$

Summing up over k and noting that the LHS telescopes:

$$f(x_{k+1}) - f(x_0) \le -\frac{1}{2L}\sum_{i=0}^{k}\|G_L(x_i)\|_2^2.$$

If $\bar{f} := \inf_{x\in\mathcal{X}} f(x) > -\infty$, then

$$\frac{1}{2L}\sum_{i=0}^{k}\|G_L(x_k)\|_2^2 \le f(x_0) - \bar{f}.$$

Hence

$$\min_{0\le i\le k}\|G_L(x_i)\|_2 \le \sqrt{\frac{2L\left(f(x_0) - \bar{f}\right)}{k+1}}.$$

Equivalently, after at most $k = \frac{8L\left(f(x_0)-\bar{f}\right)}{\epsilon^2}$ iterations of PGD, we have

$$\begin{aligned}
&\min_{0\le i\le k}\|G_L(x_i)\|_2 \le \frac{\epsilon}{2} \\
\implies &\exists i \in \{1,\dots,k+1\} : -\nabla f(x_i) \in N_{\mathcal{X}}(x_i) + \mathcal{B}_2(0,\epsilon)
\end{aligned}$$

where the last line follows from Lemma 2.

2.2 Convex case

Suppose f is L-smooth and convex, with a global minimizer x^*.

1) From HW 4: $\|G_L(x_k)\|_2 \leq \|G_L(x_{k-1})\|_2, \forall k$. (In HW3 we proved a similar monotonicity property for the gradient.) The result above thus implies

$$\|G_L(x_k)\|_2 \leq \sqrt{\frac{2L\left(f(x_0) - \bar{f}\right)}{k+1}}.$$

2) From Descent Lemma 3:

$$f(x_{k+1}) \leq f(x_k) - \frac{1}{2L}\|G_L(x_k)\|_2^2 \leq f(x_k),$$

so the function value is non-increasing in k.

3) Convexity gives the lower bound

$$f(x^*) \geq f(x_k) + \langle \nabla f(x_k), x^* - x_k \rangle,$$

whence

$$\begin{aligned} f(x_{k+1}) - f(x^*) &\leq f(x_{k+1}) - f(x_k) - \langle \nabla f(x_k), x^* - x_k \rangle \\ &= f(x_{k+1}) - f(x_k) - \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \langle \nabla f(x_k), x_{k+1} - x^* \rangle. \end{aligned} \tag{3}$$

(In the analysis of GD, we then used $\nabla f(x_k) = L(x_k - x_{k+1})$ and the 3-point identity). Recall that

$$x_{k+1} = \operatorname*{argmin}_{y \in \mathcal{X}} \left\{ \langle \nabla f(x_k), y - x_k \rangle + \frac{L}{2}\|y - x_k\|_2^2 \right\}.$$

The first-order optimality condition gives

$$\forall y \in \mathcal{X} : \langle \nabla f(x_k) + L(x_{k+1} - x_k), y - x_{k+1} \rangle \geq 0.$$

Taking $y = x^*$ gives

$$\begin{aligned} \langle \nabla f(x_k), x_{k+1} - x^* \rangle &\leq L\langle x_{k+1} - x_k, x^* - x_{k+1} \rangle \\ &= \frac{L}{2}\left(\|x_k - x^*\|_2^2 - \|x_{k+1} - x^*\|_2^2 - \|x_{k+1} - x_k\|_2^2\right). \qquad \text{3-point identity} \end{aligned}$$

Plugging into (3), we get

$$\begin{aligned} f(x_{k+1}) - f(x^*) &\leq \underbrace{f(x_{k+1}) - f(x_k) - \langle \nabla f(x_k), x_{k+1} - x_k \rangle - \frac{L}{2}\|x_{k+1} - x_k\|_2^2}_{\leq 0 \text{ by } L\text{-smoothness}} + \frac{L}{2}\|x_k - x^*\|_2^2 - \frac{L}{2}\|x_{k+1} - x^*\|_2^2 \\ &\leq \frac{L}{2}\|x_k - x^*\|_2^2 - \frac{L}{2}\|x_{k+1} - x^*\|_2^2. \end{aligned}$$

We then follow the same steps as in the analysis of GD, summing up and telescoping the above inequality:

$$\sum_{i=0}^{k} \left(f(x_{i+1}) - f(x^*)\right) \leq \frac{L}{2}\|x_0 - x^*\|_2^2 - \frac{L}{2}\|x_{k+1} - x^*\|_2^2 \leq \frac{L}{2}\|x_0 - x^*\|_2^2.$$

But LHS $\geq (k+1)\left(f(x_{k+1}) - f(x^*)\right)$ due to monotonicity $f(x_{k+1}) \leq f(x_k) \leq \cdots \leq f(x_0)$. It follows that

$$f(x_{k+1}) - f(x^*) \leq \frac{L\|x_0 - x^*\|_2^2}{2(k+1)}.$$

2.3 Strongly convex case

Suppose f is m-strongly convex and L-smooth, with a unique global minimizer x^*.

Since x^* satisfies the first-order optimality condition, we have $P_{\mathcal{X}}\left(x^* - \frac{1}{L}\nabla f(x^*)\right) = x^*$ (Lemma 1). By nonexpansiveness of $P_{\mathcal{X}}$, we have

$$\begin{aligned}\|x_{k+1} - x^*\|_2^2 &= \left\|P_{\mathcal{X}}\left(x_k - \frac{1}{L}\nabla f(x_k)\right) - P_{\mathcal{X}}\left(x^* - \frac{1}{L}\nabla f(x^*)\right)\right\|_2^2 \\ &\le \left\|\left(x_k - \frac{1}{L}\nabla f(x_k)\right) - \left(x^* - \frac{1}{L}\nabla f(x^*)\right)\right\|_2^2 \\ &= \|x_k - x^*\|_2^2 + \frac{1}{L^2}\|\nabla f(x_k) - \nabla f(x^*)\|_2^2 - \frac{2}{L}\langle x_k - x^*, \nabla f(x_k) - \nabla f(x^*)\rangle\,.\end{aligned}$$

The last RHS term satisfies the co-coercivity property

$$\|\nabla f(x_k) - \nabla f(x^*)\|_2^2 \le L\langle \nabla f(x_k) - \nabla f(x^*), x_k - x^*\rangle$$

by HW2 Q1, hence

$$\|x_{k+1} - x^*\|_2^2 \le \|x_k - x^*\|_2^2 - \frac{1}{L}\langle x_k - x^*, \nabla f(x_k) - \nabla f(x^*)\rangle\,. \tag{4}$$

By strong convexity of f:

$$\begin{aligned}f(x_k) &\ge f(x^*) + \langle \nabla f(x^*), x_k - x^*\rangle + \frac{m}{2}\|x_k - x^*\|_2^2\,, \\ f(x^*) &\ge f(x_k) + \langle \nabla f(x_k), x^* - x_k\rangle + \frac{m}{2}\|x_k - x^*\|_2^2\,.\end{aligned}$$

Adding up the two inequalities gives

$$\langle \nabla f(x_k) - \nabla f(x^*), x_k - x^*\rangle \ge m\|x_k - x^*\|_2^2\,.$$

(this is called the *strong monotonicity* or *coercivity* property of the gradient.) Plugging into (4), we obtain

$$\begin{aligned}&\|x_{k+1} - x^*\|_2^2 \le \left(1 - \frac{m}{L}\right)\|x_k - x^*\|_2^2 \\ \implies &\|x_{k+1} - x^*\|_2^2 \le \left(1 - \frac{m}{L}\right)^{k+1}\|x_0 - x^*\|_2^2\,.\end{aligned}$$

Exercise 1. Generalize the above results to PGD with a general stepsize $\frac{1}{\eta}$, where $\eta \ge L$.

3 Extensions

3.1 Acceleration (optional)

Nesterov's acceleration scheme can be extended to PGD:

$$\begin{aligned}y_k &= x_k + \beta_k(x_k - x_{k-1})\,, && \text{momentum step} \\ x_{k+1} &= P_{\mathcal{X}}(y_k - \alpha_k\nabla f(y_k))\,. && \text{projected gradient step}\end{aligned}$$

This is a special case of the *accelerated proximal gradient method* (a.k.a. fast iterative shrinkage-thresholding algorithm, FISTA), which applies to problems of the form

$$\min_{x\in\mathbb{R}^d} f(x) + g(x), \tag{5}$$

where $f : \mathbb{R}^d \to \mathbb{R}$ is convex and smooth, and $g : \mathbb{R}^d \to \bar{\mathbb{R}}$ is convex and lower semicontinuous with a computable proximal operator. Equation (5) is called a *composite problem*. As discussed in Lecture 1–2, the constrained problem (P) corresponds to a special case of the composite problem (5) with $g(x) = I_{\mathcal{X}}(x)$ being the indicator function of $\mathcal{X}$.

For details see the chapter from Beck's book.

3.2 Other search direction?

Recall that for unconstrained problems, we may use some other search direction p_k instead of the negative gradient direction and still guarantee descent in function value (Lecture 7–8).

For constrained problem, can we use some other direction $p_k \neq -\nabla f(x_k)$ in the update $x_{k+1} = P_{\mathcal{X}}\left(x_k + \frac{1}{\eta} p_k\right)$? In general, doing so does *not* guarantee the descent property $f(x_{k+1}) < f(x_k)$, even when p_k satisfies $\langle p_k, -\nabla f(x_k)\rangle > 0$. See below for an illustration.