In this lecture, we state the Matrix Bernstein inequality and sketch a few interesting applications.

References:
- The proof for the Matrix Bernstein inequality can be found in Vershynin’s book [2, Chapter 5.4]. Also see Chapter 6.6 therein.
- An exposition of the matrix Chernoff method can be found in Tropp’s paper [1], along with bounds extending beyond the case of rectangular bounded matrices.
- Also related: Wainwright’s book [3, Chapter 6.4]

1 Matrix concentration inequalities

The general idea: write a random matrix $X$ as the sum of “simple” random matrices $\sum_i X^{(i)}$.

**Theorem 1** (Matrix Bernstein inequality). Suppose that $X^{(1)}, \ldots, X^{(n)} \in \mathbb{R}^{m_1 \times m_2}$ are independent, zero-mean random matrices with

$$\|X^{(i)}\|_{op} \leq b \text{ a.s., \ \ max \ \ \left\{ \left\| \sum_i E\left( X^{(i)\top} X^{(i)} \right) \right\|_{op}, \left\| \sum_i E\left( X^{(i)} X^{(i)\top} \right) \right\|_{op} \right\} \leq \sigma^2. $$

Then we have

$$\mathbb{P}\left( \left\| \sum_i X^{(i)} \right\|_{op} \geq t \right) \leq (m_1 + m_2) \exp\left( -c \min\left\{ \frac{t^2}{\sigma^2}, \frac{tb}{b} \right\} \right). \tag{1}$$

**Remark.** In the 1-dimensional case, the quantity $\sigma^2$ reduces to the sum of variances of each element.

The proof proceeds quite naturally by mimicking that of the scalar Bernstein inequality, with one important difference: in the scalar case, we have $E(e^{X+Y}) = E(e^X)E(e^Y)$. This is no longer true generically in the matrix world, because matrices are not commutative in general. However, we can use the Golden-Thompson inequality instead:

$$\text{tr}(e^{X+Y}) \leq \text{tr}(e^X \cdot e^Y)$$

Alternatively, one may use the Lieb’s theorem, as is done in [1]

2 Applications

**Example 1** (Matrices with independent entries). Suppose $Y \in \mathbb{R}^{n \times n}$ with $Y_{ij} = \varepsilon_{ij}^{i.i.d.} \text{ Unif}\{-1,1\}$. We write $Y$ as

$$Y = \sum_{i,j} \varepsilon_{ij} e_i e_j^\top = \sum_{i,j} Y^{(i,j)}.$$

To apply Theorem 1, we need to check the conditions. Verifying the first condition is easy since $\|Y^{(i,j)}\|_{op} = 1$, implying $b = 1$. On the other hand

$$E\left( (Y^{(i,j)}\top Y^{(i,j)}) \right) = E\left( \varepsilon_{i,j}^2 e_i e_j^\top e_i e_j^\top \right) = e_j e_j^\top \Rightarrow \sum_{i,j} E\left( (Y^{(i,j)}\top Y^{(i,j)}) \right) = nI_n \tag{2}$$
The same argument applies to the other sum, hence $\sigma^2 = n$. By Matrix Bernstein with $t := C\sqrt{n \log n}$, we have

$$
P \left( \|Y\|_{op} \geq C\sqrt{n \log n} \right) \leq 2n \exp \left( -c \min \left\{ \frac{c \log n}{p}, \frac{\sqrt{n \log n}}{1} \right\} \right) = 2n \exp (-c \log n) = 2n^{-c'} \tag{3}
$$

for some constant $c' > 0$. Therefore, with probability at least $1 - 2n^{-c'}$, we get $\|Y\|_{op} \lesssim \sqrt{n \log n}$. Notice that we are loose by a factor of $\sqrt{\log n}$ compared to the bounds derived for Gaussian random matrices.

**Example 2** (Matrix Completion). Suppose we are given $Y^* \in \mathbb{R}^{n \times n}$ with $\text{rank}(Y^*) = r$ and $|Y^*_{ij}| \leq 1, \forall i, j$.

Observe:

$$
Y_{ij} = \begin{cases} Y^*_{ij}, & \text{w.p. } p \\ 0, & \text{w.p. } 1 - p \end{cases} \tag{4}
$$

Moreover, assume that $p \ll 1$. It’s easy to check that $\mathbb{E}(Y_{ij}) = pY^*_{ij}$. This is a special case of the matrix estimation problem, so we’ll take

$$
\hat{Y} := \arg\min_{Z, \text{rank}(Z) \leq r} \left\| \frac{1}{p}Y - Z \right\|_F \tag{5}
$$

To bound the error of this estimator (in Frobenius norm), we can write

$$
\frac{1}{n^2} \left\| \hat{Y} - Y^* \right\|_F^2 \lesssim \frac{r}{n^2} \left\| \frac{1}{p}Y - \hat{Y}^* \right\|_{op}^2 . \tag{6}
$$

This reduces to bounding the operator norm of random matrix with independent, zero mean entries. Define

$$
Z_{ij} \triangleq \frac{1}{p}Y_{ij} - Y^*_{ij} \implies |Z_{ij}| \leq \max \left\{ \left| \frac{Y^*_{ij}}{p} - Y^*_{ij} \right|, |Y^*_{ij}| \right\} \leq \frac{1}{p}
$$

Thus $Z_{ij}$ is subgaussian with parameter $\frac{1}{p^2}$, so using the results of last week we obtain $\|Z\|_{op} \lesssim \sqrt{n} \cdot \frac{1}{p} \implies \frac{1}{n^2} \left\| \hat{Y} - Y^* \right\|_F^2 \lesssim \frac{r}{p^2}$. Let us try the Matrix Bernstein inequality:

**Lemma 1** (Homework). For the matrices defined above, we have

$$
\left\| \frac{1}{p}Y - Y^* \right\|_{op} \lesssim \sqrt{\frac{n \log n}{p} + \frac{\log n}{p}}
$$

with probability at least $1 - 2n^{-c}$.

Using Lemma 1, we have reduced the dependence to $p$ from $\frac{1}{p}$ to $\frac{1}{\sqrt{p}}$, which is desirable since we are interested in $p \to 0$. Simplifying we get

$$
\frac{1}{n^2} \left\| \hat{Y} - Y^* \right\|_F^2 \lesssim \frac{r \log n}{np} + \frac{r \log^2 n}{n^2 p^2} .
$$

Observe that we can set $p$ as low as $\frac{r \log n}{n \varepsilon^2}$, for some $\varepsilon \in (0,1]$ and still satisfy $\frac{1}{n^2} \left\| \hat{Y} - Y^* \right\|_F^2 \lesssim \varepsilon^2$.

**Example 3** (Preference matrix completion). This problem can be posed as a ranking problem from pairwise comparisons. The setup follows:

- suppose we have $n$ teams with unknown ranking (assume such a ranking exists).
- if team $i$ is better than team $j$, we have

$$
P(i \text{ beats } k) \geq P(j \text{ beats } k), \quad \forall k.
$$
• a match is played between teams $i$ and $j$ with probability $p$. Let the probability of $i$ beating $j$ be $Y^*_ij$.

• we observe

$$Y_{ij} = \begin{cases} 1, \text{ w.p. } pY^*_ij, \\ 0, \text{ w.p. } p(1 - Y^*_ij) \\ 0, \text{ w.p. } (1 - p) \end{cases}$$ (7)

The first case in (7) corresponds to $i$ beating $j$, the second case is $j$ beating $i$, and the third case happens when no game was played.

For simplicity, let us assume that everything is independent across $(i, j)$ (i.e., when teams are matched, they play two games independently — one of them is $Y_{ij}$, and the other is $Y_{ji}$). The goal is again to recover $Y^* \in [0, 1]^{n \times n}$ given the observation $Y$. The important difference between this example and Example 2 is that $Y^*$ here is not exactly low rank.

**Estimator**: we still set $\hat{Y}$ equal to the best rank-$r$ approximation of $\frac{1}{p}Y$, as in Eq. (5). The following Claim can guide us in setting the target rank $r$:

**Claim 1.** The matrix $Y^*$ is approximately low-rank with $r = \sqrt{pm}$, i.e., there exists $Z$ with rank$(Z) \leq r$ such that

$$\begin{cases} \|Z - Y^*\|_F^2 \leq \frac{n^2}{r} \\ |Z_{ij}| \leq 1, \ \forall (i,j). \end{cases}$$ (8)

Let $\|\cdot\|_*$ denote the nuclear/trace norm of a matrix (i.e., the sum of its singular values). Since $\hat{Y}$ is the closest rank-$r$ approximation to $\frac{1}{p}Y$, we can show that

$$\|Z - \frac{1}{p}Y\|_F^2 = \|\frac{1}{p}Y - \hat{Y}\|_F^2 + \|Z - \hat{Y}\|_F^2 + 2 \langle \frac{1}{p}Y - Z, Z - \hat{Y} \rangle$$ (9)

$$\Rightarrow \|Z - \hat{Y}\|_F^2 \leq 2 \left( \frac{1}{p}Y - Z, \hat{Y} - Z \right) = 2 \left( \frac{1}{p}Y - Y^*, \hat{Y} - Z \right) + 2 \langle Y^* - Z, \hat{Y} - Z \rangle$$ (10)

$$\leq 2 \|\frac{1}{p}Y - Y^*\|_{op} \|\hat{Y} - Z\|_* + 2 \|\hat{Y} - Z\|_F \cdot \|Y^* - Z\|_F ,$$ (11)

where the penultimate inequality follows by adding and subtracting $Y^*$ in the inner product, and the last inequality is the trace Hölder inequality + Cauchy-Schwarz inequality. Since the matrix $\hat{Y} - Y^*$ has rank at most $2r$, we have that

$$\|\hat{Y} - Z\|_* \leq 2\sqrt{2r} \|\hat{Y} - Z\|_F \Rightarrow \|Z - \hat{Y}\|_F \leq \sqrt{r} \|\frac{1}{p}Y - Y^*\|_{op} + \|Y^* - Z\|_F .$$

From Lemma 1, we can readily bound the first term, while the second term is upper bounded by $\frac{n}{\sqrt{r}}$ using Claim 1 above. Now, we rewrite

$$\|\hat{Y} - Y^*\|_F \leq \|Z - Y^*\|_F + \|\hat{Y} - Z\|_F \lesssim \frac{2n}{\sqrt{r}} + \sqrt{\frac{rn \log n}{p}} .$$

Plugging in $r = \sqrt{pm}$ and dividing by $n^2$ yields the error bound

$$\frac{1}{n^2} \|\hat{Y} - Y^*\|_F^2 \lesssim \frac{\log n}{\sqrt{pm}} .$$

In particular, the error goes to zero when $p = \omega \left( \frac{\log^2 n}{n} \right)$.
Proof of Claim 1. Let us introduce some notation. Define

\[ S_i := \sum_{j=1}^{n} Y_{ij}^* \quad i = 1, \ldots, n, \]

\[ T_\ell := \left\{ i \left| S_i \in \left[ \frac{n(\ell - 1)}{r}, \frac{n\ell}{r} \right] \right. \right\}, \quad \ell = 1, \ldots, r. \]

and let \( k(\ell) := \) first element in \( T_\ell \), which we will treat as the “representative” element. For all \( i \in T_\ell \), define the \( i \)th row of our candidate low-rank matrix \( Z \) as

\[ Z_{i,:} = Y_{k(\ell),i}^* \]

which is a “discretization” of the rows of \( Y^* \). Then \( Z \in [0,1]^{n \times n} \) has at most \( r \) distinct rows, and \( \text{rank}(Z) \leq r \). Moreover, for each \( i \in T_\ell \):

- if team \( i \) is better than team \( k(\ell) \), we can write

  \[ \sum_{j=1}^{n} (Y_{ij}^* - Z_{ij})^2 = \sum_{j=1}^{n} (Y_{ij}^* - Y_{k(\ell)j}^*)^2 \leq \sum_{j=1}^{n} |Y_{ij}^* - Y_{k(\ell)j}^*| \]

  \[ \leq \sum_{j=1}^{n} (Y_{ij}^* - Y_{k(\ell)j}^*) = S_i - S_{k(\ell)} \]

  \[ \leq \frac{n}{r}, \]

  where \((\ddagger)\) is due to the fact that each element in the sum is in \([0,1]\) and \((\ddagger)\) is due to the fact that \( Y_{ij}^* > Y_{k(\ell)j}^* \) if team \( i \) beats team \( k(\ell) \). The last inequality, \((\ddagger)\), is due to the fact that the difference between row sums of elements in \( T_\ell \) is upper bounded by the width of that interval, \( \frac{n}{r} \).

- if \( k(\ell) \) beats team \( i \), then the same argument applies leading to the difference \( S_{k(\ell)} - S_i \) which is also upper bounded by \( \frac{n}{r} \).

Combining, we obtain

\[ \|Y^* - Z\|_F^2 = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} (Y_{ij}^* - Z_{ij})^2 \right) \leq n \cdot \frac{n}{r} = \frac{n^2}{r}. \]

References

