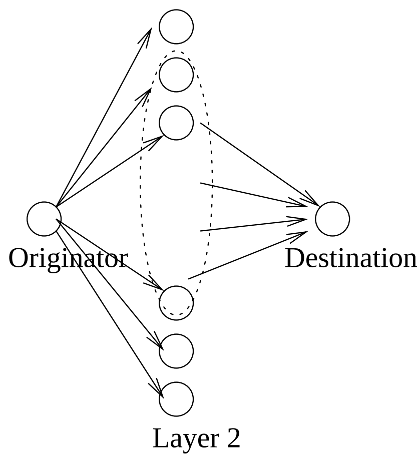
Randomized Lower Bound for Sending Message in $\mathcal{F}_n$ Graph

October 29, 2002

1 Prelimiaries

$\mathcal{F}_n$ **Graph** : Directed grpah $\mathcal{F}_n = (V, E)$ consists of $n + 2$ nodes located in 3 layers. One node called *originator*, one node called *destination* locate in 1^{st} and 3^{rd} layer respectively. The other n points labeled from 1 to n, locate in the 2^{nd} layer. There are n edges from originator to each point in layer 2 and some edges from layer 2 to destination.



Remark : The nodes of the graph represent processors and an edge from u to v implies that u can send message to v.

Time Mesurement : The time is viewed as divided into slots (or *rounds*). In any given slot, a node can either transmit some message or not.

2 Yao's Lemma

Lemma The randomized complexity $R(f)$, if both the set of inputs and the set of algorithm is finite, is euqal to the maximum over all probability distributions on inputs of the minimum expected cost of any deterministic algorithm.

Remark So in order to find the randomized lower bound in sending message in $\mathcal{F}_n$ graph, it sufficec to find a lower bound on such expected cost of any determinitic algorithm. Such a lower bound can be found by postulating an input distribution $\mathcal{P}$ and see whether every algorithm has expected cost at least equal to the desired lower bound.

3 Randomized Lower Bound $\Omega(logn)$

Deterministic Algorithm For any determinisic algorithm Π, it consists of a family of set $\{S_1,\ S_2,\ \cdots\ S_m\}$.Each set S_i is a subset of $U = \{1,\ 2,\ \cdots n\}$ which means that at i time slot, all the nodes in S_i will transmit message while other keep silence. In addition m is the running time of this algorithm.

Input Distribution Since we do not know which node connect to the destination, we use $X \subseteq U$ denoting the edges between layer 2 and destination $(Note\ \ that\ \ X \neq \phi\ )$. Suppose the probability of $|X| = k$ is p_k.

Success Transmission In some step j of Π, $|S_j \cap X| = 1$

My Reasoning Clue If I can proof that under some distribution $\mathcal{P}$, for each deterministic algorithm Π_ℓ, its running time m_ℓ for success sending message is lower bounded by $\Omega(logn)$. Then by the description above, I will conclude that the randomized lower bound for this task is $\Omega(logn)$.

I got the idea from the paper *Fast Broadcasting and Gossiping in Radio Networks*,where the author give an amortized progress is $\Omega(logn)$ then the running time of broadcasting is $O(nlog)$.

So in this case, I only need to prove that in any step the probability of success transmission is $O(\frac{1}{logn})$.Then the deterministic algorithm should take $\Omega(logn)$ round to complete the mission.

Analysis:

Suppose we have an input distribution $\mathcal{P}$, the probability $|X| = k$ is p_k.

With out lose of generality, I take the i^{th} step of a deterministic algorithm $\Pi = S_1, S_2, \cdots, S_m$.If $|X| = k$, then the probability of success transmission in is

$$p_k \cdot \frac{j \cdot \binom{n-j}{k-1}}{\binom{n}{k}}$$

where $j = |S_i|$ and the expectation is

$$\sum_{k=1}^{\ell} p_k \cdot \frac{j \cdot \binom{n-j}{k-1}}{\binom{n}{k}}$$

where $\ell = n - j + 1$ because of $n - j \geq k - 1$

So the remaining thing is to find a specific distribution s.t.

$$\sum_{k=1}^{n-j+1} p_k \cdot \frac{j \cdot \binom{n-j}{k-1}}{\binom{n}{k}} = O(\frac{1}{logn})$$

I have tried several distribution.

Naturally suppose

$$p_k = \frac{\binom{n}{k}}{\sum_{\ell=1}^{n} \binom{n}{\ell}}$$

or Uniform distribution

$$p_k = \frac{1}{n}$$

My Question

1. Whether is my reduction which using Yao's Lemma correct?
2. How to find a desired distrubition $\mathcal{P}$

Hinted By Prof. Siu-Wing Cheng Since we need find a distribution $\mathcal{P}$ to satisfy

$$\sum_{k=1}^{n-j+1} p_k \cdot \frac{j \cdot \binom{n-j}{k-1}}{\binom{n}{k}} = O(\frac{1}{logn})$$

and the right $\frac{1}{logn}$ is quite hard to eliminated by summation. So I guess

$$p_k = \frac{1}{logn} \cdot q_k$$

where $\sum_{k=1}^{n} q_k = logn$. Recall that

$$\sum_{i=1}^{\infty} \frac{1}{i} \longrightarrow logn$$

So Let $q_k = \frac{1}{k}$,what remained I should do is to prove

$$\sum_{k=1}^{n-j+1} \frac{1}{k} \cdot \frac{j \cdot \binom{n-j}{k-1}}{\binom{n}{k}} = O(1)$$

that is equivalent to prove

$$\sum_{k=1}^{n-j+1} \frac{1}{k} \cdot \frac{\binom{n-j}{k-1}}{\binom{n}{k}} = O(\frac{1}{j})$$

And let

$$\Phi(k) = \frac{1}{k} \cdot \frac{\binom{n-j}{k-1}}{\binom{n}{k}} \;\; obviously \;\; \Phi(k) > 0$$

Easily

$$\frac{\Phi(k+1)}{\Phi(k)} = \frac{n-j-k+1}{n-k} \leq \frac{n-j+1}{n} \leq 1$$

for $j \geq 1$ and$k \geq 1$.

So, we can get

$$\Phi(k) \leq \Phi(1) \cdot \hbar^{k-1}$$

where $\hbar = \frac{n-j+1}{n} \leq 1$.And

$$\sum_{k=1}^{n-j+1} \Phi(k) \leq \sum_{k=1}^{n-j+1} \Phi(1) \cdot \hbar^{k-1} \leq \sum_{k=1}^{\infty} \Phi(1) \cdot \hbar^{k-1}$$

where $\hbar = \frac{n-j+1}{n} \leq 1$. Because $\Phi(1) = \frac{1}{n}$

$$\sum_{k=1}^{\infty} \Phi(1) \cdot \hbar^{k-1} = \frac{\Phi(1)}{1-\hbar} = \frac{1}{j-1}$$

and

$$\frac{1}{j-1} \leq 3 \cdot \frac{1}{i} \in O(\frac{1}{j})$$

Q.E.D.