Randomized Lower Bound for Sending Message in $\mathcal{F}_n$ Graph

Li Yunpeng

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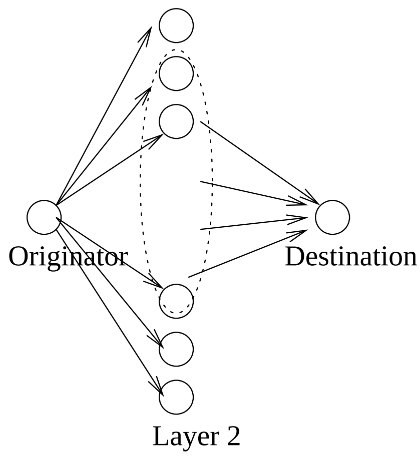
1 Prelimiaries

$\mathcal{F}_n$ Graph: Directed grpah $\mathcal{F}_n = (V, E)$ consists of $n + 2$ nodes located in 3 layers. One node called *originator*, one node called *destination* locate in 1^{st} and 3^{rd} layer respectively. The other n points labeled from 1 to n, locate in the 2^{nd} layer. There are n edges from originator to each point in layer 2 and some edges from layer 2 to destination.



Remark: The nodes of the graph represent processors and an edge from u to v implies that u can send message to v.

Time Mesurement: The time is viewed as divided into slots (or *rounds*). In any given slot, a node can either transmit some message or not.

Radio Network Property: One node can receive message if and only if exactly one of the node's neighbours transmits at time t .

Task: Send messgae from originator to destination.

2 Randomized Algorithm

- Each node P_i uses a probabilistic procedure Φ_i^j to decide whether the node will transmit at the next time slot j or not, which may depend on the *history*
- The program of P_i is a collection $\Phi_i = (\Phi_i^1, \Phi_i^2, \Phi_i^3, ...)$
- A randomized algorithm for the task in F_n is simply a collection of n such program and an initial transformation which the originator send message to all processor in layer 2.

Remark: An algorithm is *uniform* if all processors use the same program. Otherwise, the algorithm is called *nonuniform*.

3 Lower Bound $\Omega(log(n))$

3.1 Uniform Case

For an algorithm Π, we call a round *successful* if exactly one processor which is connected to destination transmits. Let $E(T_\ell^\Pi$ denote the expected number of rounds until the *first* successful round, given that there are 2^ℓ edges incident to destination.

LEMMA: Let Π be an algorithm in graph $\mathcal{F}_n$ to transmit message from originator to destination. Then

$$E_\ell[E(T_\ell^\Pi)] = \Omega(log(n)),$$

where E_ℓ denotes the expection when ℓ is chosen uniformly from the range $1 \leq \ell \leq logn$

Ideas:

- The lemma only deals with the *first* success. This, in a sense, allow us to get rid of the dependency in the history.
- Intuitively the algorithm cannot have high probability of success in (a fixed) round s for more than a few values of 2^ℓ.This would imply that since topology between layer 2 and layter 3 is unknown, $\Omega(logn)$ rounds would be required to reach a success for all possible cases.

Proof

Define $p_{s,\ell} = Pr[$ failure in rounds 1,...$s-1$ and success in round s | there is 2^ℓ edges incident to destination]

As the events described in the definition are disjoint and assuming that the algorithm succeeds with probability 1, we get

$$\sum_{s=1}^{\infty} p_{s,\ell} = 1$$

We can get rid of the dependency simply by writing:

$$p_{s,\ell} \leq Pr[success\ \ in\ \ round\ \ s|2^{\ell}\ edges\ \ to\ \ destination]$$

Formally, we make the following claim to illustrate the second idea.

CLAIM. For any s,

$$\sum_{\ell=1}^{logn} Pr[success\ \ in\ \ round\ \ s\ \ |2^{\ell}edges\ \ to\ \ destination] < 2$$

Proof:

Define $q(s) = Pr[trying\ \ in\ \ round\ \ s]$.Note that $q(s)$ does not depend on ℓ. Recall that a successful round is one in which exactly one node is trying to transmit. Therefore

$$\text{Pr[success in rouns } s\ |2^{\ell}] = 2^{\ell} \cdot q(s) \cdot (1 - q(s))^{2^{\ell}-1}$$

We get

$$\begin{aligned}
\sum_{\ell=1}^{logn} Pr[success\ \ in\ \ round\ \ s|2^{\ell}] &= \sum_{\ell=1}^{logn} 2^{\ell} q(s)(1 - q(s))^{2^{\ell}-1} \\
&= q(s) \sum_{\ell=1}^{logn} 2^{\ell}(1 - q(s))^{2^{\ell}-1} \\
&\leq 2 \cdot q(s) \sum_{j=1}^{n-1} (1 - q(s))^{j} \\
&= 2 \cdot q(s) \cdot \frac{1 - (1 - q(s))^{n}}{q(s)} < 2
\end{aligned}$$

which completes the proof of the claim. *Q.E.D*

We are interested in $\sum_{s=1}^{k} p_{s,k}$, which is intuitively the probability that, given that there are 2^{ℓ} edges incident to destination, the algorithm succeeds in one of the first k rounds.

$$\sum_{\ell=1}^{logn} \sum_{s=1}^{k} \leq \sum_{s=1}^{k} \sum_{\ell=1}^{logn} Pr[success\ \ in\ \ round\ \ s\ |2^{\ell}] < 2k$$

By definition,

$$E_\ell[E(T_\ell^\Pi)] = \sum_{\ell=1}^{logn} \frac{1}{logn} \sum_{s=1}^{\infty} p_{s,\ell} \cdot s \geq \frac{k}{logn} \sum_{\ell=1}^{logn} \sum_{s=k}^{\infty} p_{s,\ell}$$

this equals

$$\frac{k}{logn} \sum_{\ell=1}^{logn} \left(1 - \sum_{s=1}^{k-1} p_{s,\ell}\right)$$

which is greater than

$$\frac{k}{logn} \cdot (logn - 2(2(k-1)).$$

By choosing $k = \frac{1}{4}logn$ we have that

$$E_\ell[E(T_\ell^\Pi)] \geq \frac{1}{8}logn + \frac{1}{2}$$

which completes the proof of the lemma.

3.2 Nonuniform Case

3.3 Randomized Lower Bound

LEMMA: If L is accepted by a probabilistic algorithm M with 0-sided error and expected time $T(n)$, then L is accepted by a probabilistic algorithm M' with 1-sided error and worst case time $O(T(n))$.
THEOREM: The randomized lower bound for transmit message from originator to destination in graph $\mathcal{F}_n$ is $\Omega(log(n))$.