ECO2020 Tutorial 1

Young Wu

November 9, 2016

1 Question 1

1.1 Part a

Pareto Efficient allocation requires:

1. Individual feasibility:

$$x_1 \geq 0 \text{ and } x_2 \geq 0$$

2. Aggregate feasibility:

$$-2 \leq m_1 + m_2 \leq 0$$
$$\sqrt{x_1 + x_2} + m_1 + m_2 \leq 0$$

3. Optimality:

   There does not exist $((\tilde{x}_1, \tilde{m}_1), (\tilde{x}_2, \tilde{m}_2))$ such that:

$$\sqrt{\tilde{x}_1} + \tilde{m}_1 \geq \sqrt{x_1} + m_1$$
$$\sqrt{\tilde{x}_2} + \tilde{m}_2 \geq \sqrt{x_2} + m_2$$

   One of them holds with strict inequality.

Method 1 (Direct):

$$\max_{x_1,m_1} \sqrt{x_1} + m_1 \text{ such that } \sqrt{x_2} + m_2 = u \text{ and } -2 \leq m_1 + m_2 \leq 0 \text{ and } \sqrt{x_1 + x_2} + m_1 + m_2 \leq 0$$

$$\Rightarrow \max_{x_1,m_1} \sqrt{x_1} + m_1 \text{ such that } \sqrt{x_2} + m_2 = u \text{ and } m_1 + m_2 = -2 \text{ and } x_1 + x_2 = 4$$

$$\Rightarrow \max_{x_1} \sqrt{x_1} + (-2) - \left(u - \sqrt{4 - x_1}\right)$$

$$\Rightarrow \frac{1}{2\sqrt{x_1}} - \frac{1}{2\sqrt{4 - x_1}} = 0$$

$$\Rightarrow x_1 = 4 - x_1$$

$$\Rightarrow x_1 = 2$$

Then, $x_2 = 4 - x_1 = 2$ and any $m_1 + m_2 = -2$ is Pareto optimal.

There are no boundary solutions since m is not bounded.

Method 2 (MRS):

$$\text{set MRS}_1 = \text{MRS}_2$$
$$\Rightarrow \frac{1}{2\sqrt{x_1}} = \frac{1}{2\sqrt{x_2}}$$
$$\Rightarrow x_1 = x_2 = 2$$

Method 3 (Use Quasilinearity):

For quasilinear utility functions:

$$\max u_1 + u_2 \Leftrightarrow \text{ Pareto Optimal}$$

Therefore,

$$\max_{x_1,x_2,m_1,m_2} \sqrt{x_1} + m_1 + \sqrt{x_2} + m_2 \text{ such that } -2 \leq m_1 + m_2 \leq 0 \text{ and } \sqrt{x_1 + x_2} + m_1 + m_2 \leq 0$$

$$\Rightarrow \max_{x_1,x_2,m_1,m_2} \sqrt{x_1} + m_1 + \sqrt{x_2} + m_2 \text{ and } m_1 + m_2 = -2 \text{ and } x_1 + x_2 = 4$$

$$\Rightarrow \max_{x_1} \sqrt{x_1} + \sqrt{4 - x_1} - 2$$

$$\Rightarrow x_1 = 2$$

Therefore, $\{((x_1 = 2, m_1 = m), (x_2 = 2, m_2 = -2 - m), m \in \mathbb{R})\}$ are Pareto optimal.

1.2 Part b

Walrasian Equilibrium requires:

1. Consumers maximize utility:

$$\max_{x_1,m_1} \sqrt{x_1} + m_1 \text{ such that } p_x x_1 + p_m m_1 = \frac{1}{2}\pi$$
$$\max_{x_2,m_2} \sqrt{x_2} + m_2 \text{ such that } p_x x_2 + p_m m_2 = \frac{1}{2}\pi$$

2. Firm maximizes profit:

$$\pi = \max_{x,m} p_x x + p_m m \text{ such that } \sqrt{x} + m \leq 0 \text{ and } -2 \leq m \leq 0$$

3. Markets clear:

$$x_1 + x_2 = x$$
$$m_1 + m_2 = m$$

Normalize the prices to $p_m = 1$, and solve the firm's problem:

$$\max_{x,m} p_x x + m \text{ such that } \sqrt{x} + m \leq 0 \text{ and } -2 \leq m \leq 0$$
$$\Rightarrow \max_{x} p_x x - \sqrt{x} \text{ such that } 0 \leq x \leq 4$$
$$\Rightarrow \text{ FOC } : p_x - \frac{1}{2\sqrt{x}} = 0$$
$$\Rightarrow x = \frac{1}{4p_x^2}$$

Note that SOC: $\frac{1}{4}x^{-\frac{3}{2}} > 0$ is not concave. Therefore, $x \in \{0, 4\}$.

$$x = \begin{cases} 0 & \text{if } 4p_x - \sqrt{4} \leq 0p_x - \sqrt{0} \\ 4 & \text{if } 4p_x - \sqrt{4} \geq 0p_x - \sqrt{0} \end{cases}$$

Or equivalently,

$$x = \begin{cases} 0 & \text{if } p_x < \frac{1}{2} \\ 4 & \text{if } p_x > \frac{1}{2} \\ \{0, 4\} & \text{if } p_x = \frac{1}{2} \end{cases}$$

And

$$m = -\sqrt{x}$$

$$\pi = \begin{cases} 0 & \text{if } p_x < \frac{1}{2} \\ 4p_x - 2 & \text{if } p_x > \frac{1}{2} \end{cases}$$

Solve the consumer's problem:

$$\begin{aligned} &\max_{x_i, m_i} \sqrt{x_i} + m_i \text{ such that } p_x x_i + m_i = \frac{1}{2}\pi \\ &\Rightarrow \max_{x_i} \sqrt{x_i} + \frac{1}{2}\pi - p_x x_i \\ &\Rightarrow \text{ FOC } : \frac{1}{2\sqrt{x_i}} - p_x = 0 \\ &\Rightarrow x_i = \frac{1}{4p_x^2} \end{aligned}$$

Use market clearing conditions to find prices and allocations:

$$\text{set } x = x_1 + x_2 = \frac{1}{2p_x^2}$$

If $p_x \geq \frac{1}{2}$,

$$\begin{aligned} x = \frac{1}{2p_x^2} = 4 \\ \Rightarrow p_x = \sqrt{\frac{1}{8}} < \frac{1}{2} \end{aligned}$$

Contradiction.

If $p_x \leq \frac{1}{2}$,

$$\begin{aligned} x = \frac{1}{2p_x^2} = 0 \\ \Rightarrow p_x = \infty > \frac{1}{2} \end{aligned}$$

Contradiction.

Therefore, there are no price to clear the markets. No WE exist.

2 Question 2

(Comprehensive Exam June 2008 Q3)

2.1 Part a

Pareto Efficient allocation requires:

1. Individual feasibility:

$$x_A \geq 0 \text{ and } x_B \geq 0$$

$$y_A \geq 0 \text{ and } y_B \geq 0$$

2. Aggregate feasibility:

$$0 \leq x_A + x_B \leq 6$$

$$0 \leq y_A + y_B \leq 3$$

3. Optimality:

   There does not exist $((\tilde{x}_A, \tilde{y}_A), (\tilde{x}_B, \tilde{y}_B))$ such that:

$$\tilde{x}_A + \tilde{y}_A \geq x_A + y_A$$

$$\tilde{y}_B \geq y_B$$

   One of them holds with strict inequality.

Consumer B does not care about x,

$$x_A = 6, x_B = 0 \text{ and any } y_A + y_B = 3 \text{ is Pareto Optimal}$$

Therefore, $\{(x_A = 6, y_A = y), (x_B = 0, y_B = 3 - y), y \in [0, 3]\}$ are Pareto Optimal.

2.2 Part b

Walrasian Equilibrium requires:

1. Consumers maximize utility:

$$\max_{x_A, y_A} x_A + y_A \text{ such that } p_x x_A + p_y y_A = 5p_x + p_y$$

$$\max_{x_B, y_B} y_B \text{ such that } p_x x_B + p_y y_B = p_x + 2p_y$$

2. Markets clear:

$$x_A + x_B = 6$$

$$y_A + y_B = 3$$

Normalize $p_y = 1$, then consumers' problems:

$$x_A = \begin{cases} 5 + \dfrac{1}{p_x} & \text{if } p_x < 1 \\ 0 & \text{if } p_x \geq 1 \end{cases}$$

and

$$y_A = \begin{cases} 0 & \text{if } p_x < 1 \\ 5p_x + 1 & \text{if } p_x \geq 1 \end{cases}$$

Note that A wants only y if $p_x = 1$

$$x_B = 0, y_B = \frac{p_x + 2p_y}{p_y} = p_x + 2$$

Market clearing condition:

If $p_x < 1$,

$$x_A + x_B = 5 + \frac{1}{p_x} + 0 > 6$$

$x = 6$. Contradiction.

If $p_x \geq 1$,

$$y_A + y_B = 5p_x + 1 + p_x + 2 = 6p_x + 3 > 3$$

$y = 3$. Contradiction.

Therefore, there are no price to clear the markets. No WE exist.

2.3 Part c

Method 1: (Direct)

Walrasian Equilibrium requires:

1. Consumers maximize utility:

$$\max x_A + y_A \text{ such that } p_x x_A + p_y y_A = \omega_A^x p_x + \omega_A^y p_y$$

$$\max y_B \text{ such that } p_x x_B + p_y y_B = \omega_B^x p_x + \omega_B^y p_y$$

2. Markets clear:

$$x_A + x_B = 6$$

$$y_A + y_B = 3$$

Normalize $p_y = 1$, then consumers' problems:

$$x_A = \begin{cases} \omega_A^x + \dfrac{\omega_A^y}{p_x} & \text{if } p_x < 1 \\ 0 & \text{if } p_x \geq 1 \end{cases}$$

and

$$y_A = \begin{cases} 0 & \text{if } p_x < 1 \\ \omega_A^x p_x + \omega_A^y & \text{if } p_x \geq 1 \end{cases}$$

Note that A wants only y when $p_x = 1$

$$x_B = 0, y_B = \omega_B^x p_x + \omega_B^y$$

Market clearing condition:

If $p_x < 1$,

$$x_A + x_B = \omega_A^x + \frac{\omega_A^y}{p_x} = 6$$

$$y_A + y_B = \omega_B^x p_x + \omega_B^y = 3$$

$$\Rightarrow p_x = \frac{\omega_A^y}{6 - \omega_A^x} = \frac{3 - \omega_B^y}{\omega_B^x} = \frac{\omega_A^y}{\omega_B^x}$$

Need $p_x < 1$, meaning $\omega_A^y < \omega_B^x$.

Note that $\omega_A^y = \omega_B^x = 0$ is feasible as well.

If $p_x \geq 1$,

$$x_A + x_B = \omega_A^x + \frac{\omega_A^y}{p_x} = 6$$

$$y_A + y_B = \omega_A^x p_x + \omega_A^y + \omega_B^x p_x + \omega_B^y$$

$$= 6p_x + 3 = 3$$

No p_x satisfy both equalities.

Therefore, any feasible endowment in $\{((\omega_A^x, \omega_A^y), (\omega_B^x, \omega_B^y)) : \omega_A^y < \omega_B^x \text{ or } \omega_A^y = \omega_B^x = 0\}$ supports an WE.

Method 2: (Diagram)

Consumer B only wants good y.

Consumer A either wants all x or all y.

It cannot be WE if both A and B wants all y, so $p_x < 1$ and in any WE, $((x_A = 6, y_A = 0), (x_B = 0, y_B = 3))$.

Therefore, either start WE or have $p_x = \dfrac{\omega_A^y}{6 - \omega_A^x} < 1$, meaning any feasible endowment in $\{((\omega_A^x, \omega_A^y), (\omega_B^x, \omega_B^y)) : \omega_A^y < \omega_B^x \text{ or } \omega_A^y = \omega_B^x = 0\}$ supports an WE.

3 Question 3

The first three are free disposal activities:

			A4	A5	A6
-1	0	0	4	3	-3
0	-1	0	-3	-2	4
0	0	-1	-1	-2	-1

Walrasian Equilibrium requires:

1. Consumers maximize utility:

   Walras Law:

$$p_1 f_1 + p_2 f_2 + p_3 f_3 = 10p_3$$

2. Firm maximizes profit:

   Zero profit conditions:

$$\alpha_4 (4p_1 - 3p_2 - p_3) = 0$$
$$\alpha_5 (3p_1 - 2p_2 - 2p_3) = 0$$
$$\alpha_6 (-3p_1 + 4p_2 - p_3) = 0$$

3. Markets clear:

$$4\alpha_4 + 3\alpha_5 - 3\alpha_6 = f_1$$
$$-3\alpha_4 - 2\alpha_5 + 4\alpha_6 = f_2$$
$$10 - 1\alpha_4 - 2\alpha_5 - 1\alpha_6 = f_3$$

Try to solve the zero profit conditions simultaneously:

Equation $4 - 6 \Rightarrow 7p_1 - 7p_2 = 0 \Rightarrow p_1 = p_2$

Equation $4 \Rightarrow p_1 - p_3 = 0$

Equation $5 \Rightarrow p_1 - 2p_3 = 0$

Cannot have $\alpha_4, \alpha_5, \alpha_6 > 0$ at the same time:

Normalize $p_3 = 1$ and solve three cases:

Case $1 : \alpha_4, \alpha_5 > 0, \alpha_6 = 0$

Impossible since good 2 cannot be produced with activities 4 and 5.

Case 2 : $\alpha_5, \alpha_6 > 0, \alpha_4 = 0$

$$\begin{aligned} &3p_1 - 2p_2 - 2 = 0 \\ &-3p_1 + 4p_2 - 1 = 0 \\ &\Rightarrow p_1 = \frac{5}{3}, p_2 = \frac{3}{2} \end{aligned}$$

But $4p_1 - 3p_2 - p_3 = \frac{20}{3} - \frac{9}{2} - 1 = \frac{7}{6} > 0$, implying $\alpha_4 = \infty$, contradiction.

Case 3 : $\alpha_4, \alpha_6 > 0, \alpha_5 = 0$

$$\begin{aligned} &4p_1 - 3p_2 - 1 = 0 \\ &-3p_1 + 4p_2 - 1 = 0 \\ &\Rightarrow p_1 = p_2 = 1 \end{aligned}$$

Check $3p_1 - 2p_2 - 2p_3 = 3 - 2 - 2 = -1 < 0$, feasible.

Market clearing condition with Consumer's Walras Law:

$$\begin{aligned} 4\alpha_4 - 3\alpha_6 &= f_1 \\ -3\alpha_4 + 4\alpha_6 &= f_2 \\ 10 - 1\alpha_4 - 1\alpha_6 &= f_3 \\ f_1 + f_2 + f_3 &= 10 \end{aligned}$$

Solve using the first two equations and check with the last two

$$\begin{aligned} \alpha_4 &= \frac{1}{7}(4f_1 - 3f_2) \\ \alpha_6 &= \frac{1}{7}(3f_1 + 4f_2) \end{aligned}$$

Therefore, $\left\{(p_1 = p_2 = p_3 = 1), (f_1, f_2, f_3), \left(\alpha_1 = \alpha_2 = \alpha_3 = 0, \alpha_4 = \frac{1}{7}(4f_1 - 3f_2), \alpha_5 = 0, \alpha_6 = \frac{1}{7}(3f_1 + 4f_2)\right\}\right.$ is the WE.

4 Question 4

(Comprehensive Exam June 2010 Q4)

4.1 Part a

Walrasian Equilibrium requires:

1. Consumers maximize utility:

$$\max_{x_\theta, y_\theta} \alpha x_\theta + \theta Y^D y_\theta \text{ such that } p_x x_\theta + p_y y_\theta = 1p_x + 0p_y + \pi, \ \forall\, \theta \in [0,1]$$

2. Firm maximizes profit:

   No production means

$$-cp_x + 1p_y \leq 0$$

3. Markets clear:

$$x = \int_0^1 x_\theta d\theta = 1$$
$$y = Y^D = \int_0^1 y_\theta d\theta = 0$$

Normalize $p_y = 1$, (The solution normalized $p_x = 1$). Activity 3 not used means that it generates negative profits:

$$-cp_x + 1 \leq 0$$
$$p_x \geq \frac{1}{c}$$

Consumer's problem has constraint $p_x x_\theta = p_x 1$

$$x_\theta = 1$$

Therefore, $\left\{ \left(p_x = p, p_y = 1, \text{ for } p \in \left[\frac{1}{c}, \infty\right) \right), (x_\theta = 1, y_\theta = 0, \text{ for } \theta \in [0,1]), (\alpha_1 = \alpha_2 = \alpha_3 = 0) \right\}$ is a WE.

4.2 Part b

Walrasian Equilibrium requires:

1. Consumers maximize utility:

$$\max_{x_\theta, y_\theta} \alpha x_\theta + \theta Y^D y_\theta \text{ such that } p_x x_\theta + p_y y_\theta = 1p_x + 0p_y + \pi, \ \forall \, \theta \in [0,1]$$

2. Firm maximizes profit:

   Zero profit conditions:

$$-cp_x + 1p_y = 0$$

3. Markets clear:

$$\begin{aligned} x &= \int_0^1 x_\theta d\theta \\ y = Y^D &= \int_0^1 y_\theta d\theta \end{aligned}$$

Normalize $p_y = 1$, (The solution normalized $p_x = 1$). Use the firm's zero profit condition to find p_x:

$$\begin{aligned} -cp_x + 1 &= 0 \\ p_x &= \frac{1}{c} \\ \pi &= 0 \end{aligned}$$

Then consumer's problem becomes:

$$\max_{x_\theta, y_\theta} \alpha x_\theta + \theta Y^D y_\theta \text{ such that } \frac{1}{c} x_\theta + y_\theta = \frac{1}{c}$$

$$x_\theta = \begin{cases} 0 & \text{if } \dfrac{\theta Y^D}{c} > \alpha \\ 1 & \text{if } \dfrac{\theta Y^D}{c} < \alpha \\ [0,1] & \text{if } \dfrac{\theta Y^D}{c} = \alpha \end{cases}$$

and,

$$y_\theta = \begin{cases} 0 & \text{if } \dfrac{\theta Y^D}{c} < \alpha \\ \dfrac{1}{c} & \text{if } \dfrac{\theta Y^D}{c} > \alpha \\ \left[0, \dfrac{1}{c}\right] & \text{if } \dfrac{\theta Y^D}{c} = \alpha \end{cases}$$

Market clearing condition:

Method 1 (Y market)

$$\begin{aligned} Y^D &= \int_0^1 \frac{1}{c}\mathbb{I}_{\left\{\frac{\theta Y^D}{c}>\alpha\right\}} d\theta \\ &= \int_{\frac{\alpha c}{Y^D}} \frac{1}{c} d\theta \\ &= \frac{1}{c} - \frac{\alpha}{Y^D} \end{aligned}$$

Solve for Y^D using $\alpha c^2 = \frac{1}{4}$:

$$\begin{aligned} c\left(Y^D\right)^2 - Y^D - \alpha c &= 0 \\ \Rightarrow Y^D &= \frac{1 \pm \sqrt{1 - 4\alpha c^2}}{2c} \\ \Rightarrow Y^D &= \frac{1}{2c} \end{aligned}$$

Method 2 (X market):

$$\begin{aligned} &\int_0^1 1\mathbb{I}_{\left\{\frac{\theta Y^D}{c}<\alpha\right\}} d\theta \\ &= \int_0^{\frac{\alpha c}{Y^D}} d\theta \\ &= \frac{\alpha c}{Y^D} \end{aligned}$$

$\left(1 - \frac{\alpha c}{Y^D}\right)$ units of X are converted into Y^D units of Y, therefore as before

$$\begin{aligned} \left(1 - \frac{\alpha c}{Y^D}\right) \cdot c &= Y^D \\ \Rightarrow Y^D &= \frac{1}{2c} \end{aligned}$$

In any case, the condition $\frac{\theta Y^D}{c} > \alpha$ becomes:

$$\begin{aligned} \theta \frac{1}{2c}\frac{1}{c} &> \alpha \\ \theta &> 2\alpha c^2 \\ \theta &> \frac{1}{2} \end{aligned}$$

This means $\frac{1}{2c}$ level of activity 3 is used to convert $c \cdot \frac{1}{2c} = \frac{1}{2}$ unit of X into $\frac{1}{2c}$ units of Y.

$\left\{ \left(p_x = \frac{1}{c}, p_y = 1 \right), \left(x_\theta = 1\mathbb{I}_{\left\{ \theta \leq \frac{1}{2} \right\}}, y_\theta = \frac{1}{c}\mathbb{I}_{\left\{ \theta > \frac{1}{2} \right)} \right), \text{ for } \theta \in [0,1], \left(\alpha_1 = \alpha_2 = 0, \alpha_3 = \frac{1}{2c} \right) \right\}$ is a WE.

4.3 Part c

For $\theta \leq \frac{1}{2}$, they get the same allocation $(x_\theta = 1, y_\theta = 0)$ resulting in utility α.

For $\theta > \frac{1}{2}$, they get strictly preferred allocations $\left(x_\theta = 0, y_\theta = \frac{1}{c} \right)$ resulting in utility $\theta \frac{1}{2c} \frac{1}{c} = 2\theta\alpha > \alpha$.

Allocation in Part b Pareto dominates allocation in Part c.