1 Question 1

1.1 Part a

Pareto Efficient allocation requires:

1. Individual feasibility:

\[ x_1 \geq 0 \text{ and } x_2 \geq 0 \]

2. Aggregate feasibility:

\[ -2 \leq m_1 + m_2 \leq 0 \]
\[ \sqrt{x_1 + x_2 + m_1 + m_2} \leq 0 \]

3. Optimality:

There does not exist \(((\tilde{x}_1, \tilde{m}_1), (\tilde{x}_2, \tilde{m}_2))\) such that:

\[ \sqrt{\tilde{x}_1 + \tilde{m}_1} \geq \sqrt{x_1 + m_1} \]
\[ \sqrt{\tilde{x}_2 + \tilde{m}_2} \geq \sqrt{x_2 + m_2} \]

One of them holds with strict inequality.
Method 1 (Direct):

\[
\max_{x_1, m_1} \sqrt{x_1} + m_1 \text{ such that } \sqrt{x_2} + m_2 = u \text{ and } -2 \leq m_1 + m_2 \leq 0 \text{ and } \sqrt{x_1 + x_2} + m_1 + m_2 \leq 0
\]

\[
\Rightarrow \max_{x_1, m_1} \sqrt{x_1} + m_1 \text{ such that } \sqrt{x_2} + m_2 = u \text{ and } m_1 + m_2 = -2 \text{ and } x_1 + x_2 = 4
\]

\[
\Rightarrow \max_{x_1} \sqrt{x_1} + (-2) - (u - \sqrt{4 - x_1})
\]

\[
\Rightarrow \frac{1}{2 \sqrt{x_1}} - \frac{1}{2 \sqrt{4 - x_1}} = 0
\]

\[
\Rightarrow x_1 = 4 - x_1
\]

\[
\Rightarrow x_1 = 2
\]

Then, \(x_2 = 4 - x_1 = 2\) and any \(m_1 + m_2 = -2\) is Pareto optimal.

There are no boundary solutions since \(m\) is not bounded.

Method 2 (MRS):

\[
\text{set } \text{MRS}_1 = \text{MRS}_2
\]

\[
\Rightarrow \frac{1}{2 \sqrt{x_1}} = \frac{1}{2 \sqrt{x_2}}
\]

\[
\Rightarrow x_1 = x_2 = 2
\]

Method 3 (Use Quasilinearity):

For quasilinear utility functions:

\[
\max u_1 + u_2 \Leftrightarrow \text{Pareto Optimal}
\]

Therefore,

\[
\max_{x_1, x_2, m_1, m_2} \sqrt{x_1} + m_1 + \sqrt{x_2} + m_2 \text{ such that } -2 \leq m_1 + m_2 \leq 0 \text{ and } \sqrt{x_1 + x_2} + m_1 + m_2 \leq 0
\]

\[
\Rightarrow \max_{x_1, x_2, m_1, m_2} \sqrt{x_1} + m_1 + \sqrt{x_2} + m_2 \text{ and } m_1 + m_2 = -2 \text{ and } x_1 + x_2 = 4
\]

\[
\Rightarrow \max_{x_1} \sqrt{x_1} + \sqrt{4 - x_1} - 2
\]

\[
\Rightarrow x_1 = 2
\]

Therefore, \(\{(x_1 = 2, m_1 = m), (x_2 = 2, m_2 = -2 - m), m \in \mathbb{R}\}\) are Pareto optimal.
1.2 Part b

Walrasian Equilibrium requires:

1. Consumers maximize utility:

\[
\max_{x_1, m_1} \sqrt{x_1} + m_1 \text{ such that } p_x x_1 + p_m m_1 = \frac{1}{2}
\]

\[
\max_{x_2, m_2} \sqrt{x_2} + m_2 \text{ such that } p_x x_2 + p_m m_2 = \frac{1}{2}
\]

2. Firm maximizes profit:

\[
\pi = \max_{x, m} p_x x + p_m m \text{ such that } \sqrt{x} + m \leq 0 \text{ and } -2 \leq m \leq 0
\]

3. Markets clear:

\[
x_1 + x_2 = x
\]

\[
m_1 + m_2 = m
\]

Normalize the prices to \( p_m = 1 \), and solve the firm’s problem:

\[
\max_{x, m} p_x x + m \text{ such that } \sqrt{x} + m \leq 0 \text{ and } -2 \leq m \leq 0
\]

\[\Rightarrow \max_x p_x x - \sqrt{x} \text{ such that } 0 \leq x \leq 4\]

\[\Rightarrow \text{ FOC : } p_x - \frac{1}{2 \sqrt{x}} = 0\]

\[\Rightarrow x = \frac{1}{4p_x^2}\]

Note that SOC: \( \frac{3}{4} x^{-\frac{3}{2}} > 0 \) is not concave. Therefore, \( x \in \{0, 4\} \).

\[
x = \begin{cases} 
0 & \text{if } 4p_x - \sqrt{4} \leq 0p_x - \sqrt{0} \\
4 & \text{if } 4p_x - \sqrt{4} \geq 0p_x - \sqrt{0}
\end{cases}
\]
Or equivalently,

\[ x = \begin{cases} 
0 & \text{if } p_x < \frac{1}{2} \\
4 & \text{if } p_x > \frac{1}{2} \\
\{0, 4\} & \text{if } p_x = \frac{1}{2}
\end{cases} \]

And

\[ m = -\sqrt{x} \]

\[ \pi = \begin{cases} 
0 & \text{if } p_x < \frac{1}{2} \\
4p_x - 2 & \text{if } p_x > \frac{1}{2}
\end{cases} \]

Solve the consumer’s problem:

\[
\max_{x, m_i} \sqrt{x_i} + m_i \text{ such that } p_x x_i + m_i = \frac{1}{2} \pi
\]

\[ \Rightarrow \max_{x_i} \sqrt{x_i} + \frac{1}{2} \pi - p_x x_i \]

\[ \Rightarrow \text{FOC : } \frac{1}{2\sqrt{x_i}} - p_x = 0 \]

\[ \Rightarrow x_i = \frac{1}{4p_x^2} \]

Use market clearing conditions to find prices and allocations:

set \( x = x_1 + x_2 = \frac{1}{2p_x^2} \)

If \( p_x \geq \frac{1}{2} \),

\[ x = \frac{1}{2p_x^2} = 4 \]

\[ \Rightarrow p_x = \sqrt{\frac{1}{8}} < \frac{1}{2} \]

Contradiction.

If \( p_x \leq \frac{1}{2} \),

\[ x = \frac{1}{2p_x^2} = 0 \]

\[ \Rightarrow p_x = \infty > \frac{1}{2} \]
Contradiction.
Therefore, there are no price to clear the markets. No WE exist.

2 Question 2

(Comprehensive Exam June 2008 Q3)

2.1 Part a

Pareto Efficient allocation requires:

1. Individual feasibility:

\[ x_A \geq 0 \text{ and } x_B \geq 0 \]
\[ y_A \geq 0 \text{ and } y_B \geq 0 \]

2. Aggregate feasibility:

\[ 0 \leq x_A + x_B \leq 6 \]
\[ 0 \leq y_A + y_B \leq 3 \]

3. Optimality:

There does not exist \((\tilde{x}_A, \tilde{y}_A, \tilde{x}_B, \tilde{y}_B)\) such that:

\[ \tilde{x}_A + \tilde{y}_A \geq x_A + y_A \]
\[ \tilde{y}_B \geq y_B \]

One of them holds with strict inequality.

Consumer B does not care about \(x\),

\[ x_A = 6, x_B = 0 \text{ and } \text{any } y_A + y_B = 3 \text{ is Pareto Optimal} \]

Therefore, \( \{(x_A = 6, y_A = y), (x_B = 0, y_B = 3 - y), y \in [0, 3]\} \) are Pareto Optimal.
2.2 Part b

Walrasian Equilibrium requires:

1. Consumers maximize utility:

\[ \max_{x_A, y_A} x_A + y_A \text{ such that } p_x x_A + p_y y_A = 5p_x + p_y \]

\[ \max_{x_B, y_B} y_B \text{ such that } p_x x_B + p_y y_B = p_x + 2p_y \]

2. Markets clear:

\[ x_A + x_B = 6 \]

\[ y_A + y_B = 3 \]

Normalize \( p_y = 1 \), then consumers’ problems:

\[ x_A = \begin{cases} 
5 + \frac{1}{p_x} & \text{if } p_x < 1 \\
0 & \text{if } p_x \geq 1 
\end{cases} \]

and

\[ y_A = \begin{cases} 
0 & \text{if } p_x < 1 \\
5p_x + 1 & \text{if } p_x \geq 1 
\end{cases} \]

Note that \( A \) wants only \( y \) if \( p_x = 1 \)

\[ x_B = 0, y_B = \frac{p_x + 2p_y}{p_y} = p_x + 2 \]

Market clearing condition:

If \( p_x < 1 \),

\[ x_A + x_B = 5 + \frac{1}{p_x} + 0 > 6 \]

\( x = 6 \). Contradiction.

If \( p_x \geq 1 \),

\[ y_A + y_B = 5p_x + 1 + p_x + 2 = 6p_x + 3 > 3 \]
$y = 3$. Contradiction.

Therefore, there are no price to clear the markets. No WE exist.

2.3 Part c

Method 1: (Direct)

Walrasian Equilibrium requires:

1. Consumers maximize utility:

$$\max x_A + y_A \text{ such that } p_x x_A + p_y y_A = \omega^x_A p_x + \omega^y_A p_y$$

$$\max y_B \text{ such that } p_x x_B + p_y y_B = \omega^x_B p_x + \omega^y_B p_y$$

2. Markets clear:

$$x_A + x_B = 6$$

$$y_A + y_B = 3$$

Normalize $p_y = 1$, then consumers’ problems:

$$x_A = \begin{cases} 
\omega^x_A + \frac{\omega^y_A}{p_x} & \text{if } p_x < 1 \\
0 & \text{if } p_x \geq 1 
\end{cases}$$

and

$$y_A = \begin{cases} 
0 & \text{if } p_x < 1 \\
\omega^x_A p_x + \omega^y_A & \text{if } p_x \geq 1 
\end{cases}$$

Note that $A$ wants only $y$ when $p_x = 1$

$$x_B = 0, y_B = \omega^x_B p_x + \omega^y_B$$

Market clearing condition:
If \( p_x < 1 \),

\[
\begin{align*}
x_A + x_B &= \omega^x_A + \frac{\omega^y_A}{p_x} = 6 \\
y_A + y_B &= \omega^x_B p_x + \omega^y_B = 3
\end{align*}
\]

\[\Rightarrow p_x = \frac{\omega^y_A}{6 - \omega^x_A} = \frac{3 - \omega^y_B}{\omega^x_B} = \frac{\omega^y_A}{\omega^x_B}\]

Need \( p_x < 1 \), meaning \( \omega^y_A < \omega^x_B \).

Note that \( \omega^y_A = \omega^x_B = 0 \) is feasible as well.

If \( p_x \geq 1 \),

\[
\begin{align*}
x_A + x_B &= \omega^x_A + \frac{\omega^y_A}{p_x} = 6 \\
y_A + y_B &= \omega^x_A p_x + \omega^y_A + \omega^x_B p_x + \omega^y_B \\
&= 6p_x + 3 = 3
\end{align*}
\]

No \( p_x \) satisfy both equalities.

Therefore, any feasible endowment in \( \{(\omega^x_A, \omega^y_A), (\omega^x_B, \omega^y_B)\) : \( \omega^y_A < \omega^x_B \) or \( \omega^y_A = \omega^x_B = 0 \} \) supports an WE.

Method 2: (Diagram)

Consumer B only wants good \( y \).

Consumer A either wants all \( x \) or all \( y \).

It cannot be WE if both A and B wants all \( y \), so \( p_x < 1 \) and in any WE, \( (x_A = 6, y_A = 0), (x_B = 0, y_B = 3) \).

Therefore, either start WE or have \( p_x = \frac{\omega^x_A}{6 - \omega^x_A} < 1 \), meaning any feasible endowment in

\( \{(\omega^x_A, \omega^y_A), (\omega^x_B, \omega^y_B)\) : \( \omega^y_A < \omega^x_B \) or \( \omega^y_A = \omega^x_B = 0 \} \) supports an WE.

### 3 Question 3

The first three are free disposal activities:

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</tr>
</tbody>
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8
Walrasian Equilibrium requires:

1. Consumers maximize utility:

Walras Law:

\[ p_1 f_1 + p_2 f_2 + p_3 f_3 = 10p_3 \]

2. Firm maximizes profit:

Zero profit conditions:

\[ \alpha_4 (4p_1 - 3p_2 - p_3) = 0 \]
\[ \alpha_5 (3p_1 - 2p_2 - 2p_3) = 0 \]
\[ \alpha_6 (-3p_1 + 4p_2 - p_3) = 0 \]

3. Markets clear:

\[ 4\alpha_4 + 3\alpha_5 - 3\alpha_6 = f_1 \]
\[ -3\alpha_4 - 2\alpha_5 + 4\alpha_6 = f_2 \]
\[ 10 - 1\alpha_4 - 2\alpha_5 - 1\alpha_6 = f_3 \]

Try to solve the zero profit conditions simultaneously:

Equation 4 - 6 \( \Rightarrow \) \( 7p_1 - 7p_2 = 0 \) \( \Rightarrow p_1 = p_2 \)
Equation 4 \( \Rightarrow p_1 - p_3 = 0 \)
Equation 5 \( \Rightarrow p_1 - 2p_3 = 0 \)

Cannot have \( \alpha_4, \alpha_5, \alpha_6 > 0 \) at the same time:

Normalize \( p_3 = 1 \) and solve three cases:

Case 1 : \( \alpha_4, \alpha_5 > 0, \alpha_6 = 0 \)

Impossible since good 2 cannot be produced with activities 4 and 5.
Case 2: \( \alpha_5, \alpha_6 > 0, \alpha_4 = 0 \)

\[
3p_1 - 2p_2 - 2 = 0 \\
-3p_1 + 4p_2 - 1 = 0 \\
\Rightarrow p_1 = \frac{5}{3}, p_2 = \frac{3}{2}
\]

But \( 4p_1 - 3p_2 - p_3 = \frac{20}{3} - \frac{9}{2} - 1 = \frac{7}{6} > 0 \), implying \( \alpha_4 = \infty \), contradiction.

Case 3: \( \alpha_4, \alpha_6 > 0, \alpha_5 = 0 \)

\[
4p_1 - 3p_2 - 1 = 0 \\
-3p_1 + 4p_2 - 1 = 0 \\
\Rightarrow p_1 = p_2 = 1
\]

Check \( 3p_1 - 2p_2 - 2p_3 = 3 - 2 - 2 = -1 < 0 \), feasible.

Market clearing condition with Consumer’s Walras Law:

\[
4\alpha_4 - 3\alpha_6 = f_1 \\
-3\alpha_4 + 4\alpha_6 = f_2 \\
10 - 1\alpha_4 - 1\alpha_6 = f_3 \\
f_1 + f_2 + f_3 = 10
\]

Solve using the first two equations and check with the last two

\[
\alpha_4 = \frac{1}{7} (4f_1 - 3f_2) \\
\alpha_6 = \frac{1}{7} (3f_1 + 4f_2)
\]

Therefore, \( \left\{ (p_1 = p_2 = p_3 = 1), (f_1, f_2, f_3), \left( \alpha_1 = \alpha_2 = \alpha_3 = 0, \alpha_4 = \frac{1}{7} (4f_1 - 3f_2), \alpha_5 = 0, \alpha_6 = \frac{1}{7} (3f_1 + 4f_2) \right) \right\} \) is the WE.

4 Question 4

(Comprehensive Exam June 2010 Q4)
4.1 Part a

Walrasian Equilibrium requires:

1. Consumers maximize utility:

   \[
   \max_{x_{\theta}, y_{\theta}} \alpha x_{\theta} + \theta Y_D x_{\theta} \text{ such that } p_x x_{\theta} + p_y y_{\theta} = 1 p_x + 0 p_y + \pi, \ \forall \theta \in [0,1]
   \]

2. Firm maximizes profit:

   No production means

   \[-cp_x + 1p_y \leq 0\]

3. Markets clear:

   \[
   x = \int_0^1 x_{\theta} d\theta = 1
   \]

   \[
   y = Y^D = \int_0^1 y_{\theta} d\theta = 0
   \]

   Normalize \(p_y = 1\), (The solution normalized \(p_x = 1\)). Activity 3 not used means that it generates negative profits:

   \[-cp_x + 1 \leq 0\]

   \[p_x \geq \frac{1}{c}\]

Consumer’s problem has constraint \(p_x x_{\theta} = p_x 1\)

\[x_{\theta} = 1\]

Therefore, \(\left\{ \left( p_x = p, p_y = 1, \text{ for } p \in \left[ \frac{1}{c}, \infty \right) \right), (x_{\theta} = 1, y_{\theta} = 0, \text{ for } \theta \in [0,1]), (\alpha_1 = \alpha_2 = \alpha_3 = 0) \right\} \) is a WE.

4.2 Part b

Walrasian Equilibrium requires:
1. Consumers maximize utility:

\[
\max_{x, y} \alpha x_\theta + \theta Y^D y_\theta \text{ such that } p_x x_\theta + p_y y_\theta = 1 p_x + 0 p_y + \pi, \ \forall \theta \in [0, 1]
\]

2. Firm maximizes profit:

Zero profit conditions:

\[-cp_x + 1 p_y = 0\]

3. Markets clear:

\[
x = \int_0^1 x_\theta d\theta
\]

\[
y = Y^D = \int_0^1 y_\theta d\theta
\]

Normalize \( p_y = 1 \), (The solution normalized \( p_x = 1 \)). Use the firm’s zero profit condition to find \( p_x \):

\[-cp_x + 1 = 0\]

\[p_x = \frac{1}{c}\]

\[\pi = 0\]

Then consumer’s problem becomes:

\[
\max_{x, y} \alpha x_\theta + \theta Y^D y_\theta \text{ such that } \frac{1}{c} x_\theta + y_\theta = \frac{1}{c}
\]

\[x_\theta = \begin{cases}
0 & \text{if } \frac{\theta Y^D}{c} > \alpha \\
1 & \text{if } \frac{\theta Y^D}{c} < \alpha \\
[0, 1] & \text{if } \frac{\theta Y^D}{c} = \alpha
\end{cases}\]

and,

\[y_\theta = \begin{cases}
0 & \text{if } \frac{\theta Y^D}{c} < \alpha \\
1 & \text{if } \frac{\theta Y^D}{c} > \alpha \\
\frac{c}{c} & \text{if } \frac{\theta Y^D}{c} = \alpha
\end{cases}\]

Market clearing condition:
Method 1 (Y market)

\[ Y^D = \int_0^1 \frac{1}{c} \left\{ \begin{array}{ll}
\theta Y^D \\
\frac{\theta Y^D}{c} > \alpha
\end{array} \right\} d\theta \]

\[ = \int_{\alpha c}^{1} \frac{1}{c} d\theta \]

\[ = \frac{1}{c} - \frac{\alpha}{Y^D} \]

Solve for \( Y^D \) using \( \alpha c^2 = \frac{1}{4} \):

\[ c \left( Y^D \right)^2 - Y^D - \alpha c = 0 \]

\[ \Rightarrow Y^D = \frac{1 \pm \sqrt{1 - 4\alpha c^2}}{2c} \]

\[ \Rightarrow Y^D = \frac{1}{2c} \]

Method 2 (X market):

\[ \int_0^1 \frac{1}{c} \left\{ \theta Y^D < \alpha \right\} d\theta \]

\[ = \int_0^{\alpha c} \frac{1}{Y^D} d\theta \]

\[ = \frac{\alpha c}{Y^D} \]

\[ \left( 1 - \frac{\alpha c}{Y^D} \right) \text{ units of } X \text{ are converted into } Y^D \text{ units of } Y, \text{ therefore as before} \]

\[ \left( 1 - \frac{\alpha c}{Y^D} \right) \cdot c = Y^D \]

\[ \Rightarrow Y^D = \frac{1}{2c} \]

In any case, the condition \( \frac{\theta Y^D}{c} > \alpha \) becomes:

\[ \theta \frac{1}{2c} \frac{1}{c} > \alpha \]

\[ \theta > 2\alpha c^2 \]

\[ \theta > \frac{1}{2} \]
This means $\frac{1}{2c}$ level of activity 3 is used to convert $c \cdot \frac{1}{2c} = \frac{1}{2}$ unit of X into $\frac{1}{2c}$ units of Y.

$$\left\{ \begin{array}{l}
(p_x = \frac{1}{c}, p_y = 1), \quad \left( x_{\theta} = 1 \begin{cases} \frac{1}{2} & \theta \leq \frac{1}{2} \\ \frac{\theta}{2} & \theta > \frac{1}{2} \end{cases} \\ y_{\theta} = \frac{1}{c} \end{cases}, \quad \text{for } \theta \in [0,1], \quad \left( \alpha_1 = \alpha_2 = 0, \alpha_3 = \frac{1}{2c} \right) \right\} \text{ is a WE.}
$$

4.3 Part c

For $\theta \leq \frac{1}{2}$, they get the same allocation $(x_{\theta} = 1, y_{\theta} = 0)$ resulting in utility $\alpha$.

For $\theta > \frac{1}{2}$, they get strictly preferred allocations $\left( x_{\theta} = 0, y_{\theta} = \frac{1}{c} \right)$ resulting in utility $\theta \frac{1}{2c} \frac{1}{c} = 2\theta \alpha > \alpha$.

Allocation in Part b Pareto dominates allocation in Part c.