1 Question 1

(Comprehensive Exam June 2005 Q3)

1.1 Part a

Competitive Equilibrium requires:

1. Consumers maximize utility:

\[ q_1(\alpha) \geq \theta q_2(\alpha) \text{ for } \theta \in \Theta_1 \]
\[ q_1(\alpha) \leq \theta q_2(\alpha) \text{ for } \theta \in \Theta_2 \]

2. Markets clear:

\[ \int_{\Theta_1} f(\theta) \, d\theta = \alpha \]

Consider \( \hat{\theta} = \frac{q_1(\alpha)}{q_2(\alpha)} \),

If \( \theta < \hat{\theta} \),

\[ q_1(\alpha) = q_2(\alpha) \hat{\theta} > q_2(\alpha) \theta \]
\[ \Rightarrow \theta \in \Theta_1 \]

If \( \theta > \hat{\theta} \),

\[ q_1(\alpha) = q_2(\alpha) \hat{\theta} < q_2(\alpha) \theta \]
\[ \Rightarrow \theta \in \Theta_2 \]
Therefore,

\[ \Theta_1 = [0, \hat{\theta}] \]
\[ \Theta_2 = [\hat{\theta}, 1] \]

1.2 Part b

\[ \hat{\theta} = \frac{q_1(\alpha)}{q_2}(\alpha) \]
\[ = \frac{2a + \alpha}{2a + \alpha + 1} \]

Then market clearing condition becomes:

\[ \int_{\Theta_1} f(\theta) d\theta = \alpha \]
\[ \Rightarrow \int_0^{\hat{\theta}} f(\theta) d\theta = \alpha \]
\[ \Rightarrow F(\hat{\theta}) = \alpha \]
\[ \Rightarrow F\left(\frac{2a + \alpha}{2a + \alpha + 1}\right) = \alpha \]

Let \( G(\alpha) = F\left(\frac{2a + \alpha}{2a + \alpha + 1}\right) \):

\( G \) is continuous since is composite of continuous functions on \( \alpha \in [0, 1] \).
The support is \([0, 1]\), compact convex,

Apply Brower fixed point theorem:

\[ \exists \alpha^* \text{ such that } G(\alpha^*) = \alpha^* \]

2 Question 2

(Comprehensive Exam June 2007 Q3)

2.1 Part a

Let allocations be functions \( x : [0, 1] \to \mathbb{R}_+ \) and \( y : [0, 1] \to \mathbb{R}_+ \)

Pareto Efficient allocation requires:
1. Individual feasibility:

\[ x(\theta) \geq 0 \text{ and } y(\theta) \geq 0 \text{ for } \theta \in [0,1] \]

2. Aggregate feasibility:

\[
\int_0^1 x(\theta) = 1 \\
\int_0^1 y(\theta) = z
\]

3. Optimality:

There does not exist \((\tilde{x}, \tilde{y})\) such that:

\[(\tilde{x}(\theta), \tilde{y}(\theta)) \succ_\theta (x(\theta), y(\theta)) \text{ for } \theta \in [0,1]\]

with strict inequality on some non-empty open interval \((a,b) \in [0,1]\).

### 2.2 Part b

Walrasian Equilibrium requires:

1. Consumers maximize utility:

\[
\max_{x(\theta), y(\theta)} x(\theta)^\theta y(\theta)^{1-\theta} \text{ such that } p_x x(\theta) + p_y y(\theta) = p_x + p_y z \text{ for } \theta \in [0,1]
\]

2. Markets clear:

\[
\int_0^1 x(\theta) = 1 \\
\int_0^1 y(\theta) = z
\]
2.3 Part c

Normalize \( p_y = 1 \), then

\[
x(\theta) = \theta \frac{p_x + z}{p_x}
\]
\[
y(\theta) = (1 - \theta) (p_x + z)
\]

Check X market clearing:

\[
\int_0^1 \theta \frac{p_x + z}{p_x} = 1
\]
\[
\Rightarrow \frac{1}{2} \frac{p_x + z}{p_x} = 1
\]
\[
\Rightarrow p_x = z
\]

Or check Y market clearing:

\[
\int_0^1 (1 - \theta) (p_x + z) = z
\]
\[
\Rightarrow \frac{1}{2} (p_x + z) = z
\]
\[
\Rightarrow p_x = z
\]

Therefore, WE has allocations:

\[
x(\theta) = \theta \frac{z + z}{z} = 2\theta
\]
\[
y(\theta) = (1 - \theta) (z + z) = (1 - \theta) 2z
\]

3 Question 3

(Comprehensive Exam June 2007 Q4)

3.1 Part a

Walrasian Equilibrium requires:
1. Consumers maximize utility:

\[
\begin{align*}
\max_{x_A,y_A} & \min \{2x_A,y_A\} \text{ such that } p_x x_A + p_y y_A = p_x + p_y \\
\max_{x_B,y_B} & \min \{x_B,2y_B\} \text{ such that } p_x x_B + p_y y_B = p_x + p_y \\
\max_{x_C,y_C} & \min \{2x_C,y_C\} \text{ such that } p_x x_C + p_y y_C = p_x + p_y \\
\end{align*}
\]

2. Markets clear:

\[
\begin{align*}
& x_A + x_B + x_C = 3 \\
& y_A + y_B + y_C = 3 \\
\end{align*}
\]

Maybe not normalize \( p_y = 1 \) this time:

set \( 2x_A = y_A \) and \( x_B = 2y_B \) and \( 2x_C = y_C \)

Substitute them into the budget constraints:

\[
\begin{align*}
& x_A = \frac{p_x + p_y}{p_x + 2p_y} \\
& y_A = \frac{2(p_x + p_y)}{p_x + 2p_y} \\
& x_B = \frac{2(p_x + p_y)}{2p_x + p_y} \\
& y_B = \frac{p_x + p_y}{2p_x + p_y} \\
& x_C = \frac{p_x + p_y}{p_x + 2p_y} \\
& y_C = \frac{2(p_x + p_y)}{p_x + 2p_y} \\
\end{align*}
\]

Check X market clearing:

\[
\frac{p_x + p_y}{p_x + 2p_y} + \frac{2(p_x + p_y)}{2p_x + p_y} + \frac{p_x + p_y}{p_x + 2p_y} = 3 \\
\Rightarrow \frac{2p_y}{p_x + 2p_y} = \frac{p_y}{2p_x + p_y} \\
\Rightarrow p_y = 0 \text{ or } 4p_x + 2p_y = p_x + 2p_y \\
\Rightarrow p_y = 0 \text{ or } p_x = 0 \\
\]

5
AND check Y market clearing:

\[
\frac{2(p_x + p_y)}{p_x + 2p_y} + \frac{p_x + p_y}{2p_x + p_y} + \frac{2(p_x + p_y)}{p_x + 2p_y} = 3
\]

\[
\Rightarrow \frac{2p_x}{p_x + 2p_y} = \frac{p_x}{p_x + 2p_y}
\]

\[
\Rightarrow p_x = 0 \text{ or } 2p_x + 4p_y = p_x + 2p_y
\]

\[
\Rightarrow p_x = 0
\]

Note that \(p_y = 0\) is NOT a solution here.

Therefore, WE has allocations:

\[
x_A = \frac{1}{2}, x_B = 2, x_C = \frac{1}{2}
\]

\[
y_A = 1, y_B = 1, y_C = 1
\]

### 3.2 Part b

Core allocation requires:

\[
\min \left\{ 2 \cdot \frac{1 + \varepsilon}{2}, 1 + \varepsilon \right\} \geq \min \{2x_A, y_A\}
\]

\[
\min \{2 - \varepsilon, 2 \cdot (1 - 2\varepsilon)\} \geq \min \{x_B, 2y_B\}
\]

\[
\min \left\{ 2 \cdot \frac{1 + \varepsilon}{2}, 1 + \varepsilon \right\} \geq \min \{2x_C, y_C\}
\]

1. Not blocked by \(\{A\}, \{B\}, \{C\}\):

\[
x_A = y_A = 1
\]

\[
x_B = y_B = 1
\]

\[
x_C = y_C = 1
\]

2. Not blocked by \(\{A, B\}, \{A, C\}, \{B, C\}\):

\[
x_A + x_B = y_A + y_B = 2
\]

\[
x_A + x_C = y_A + y_C = 2
\]

\[
x_B + x_C = y_B + y_C = 2
\]
3. Not blocked by \( \{A, B, C\} \):

\[
x_A + x_B + x_C = y_A + y_B + y_C = 3
\]

First simplify the conditions:

\[
1 + \varepsilon \geq \min \{2x_A, y_A\}
\]

\[
2 - 4\varepsilon \geq \min \{x_B, 2y_B\} \text{ this assumes } \varepsilon \geq 0
\]

\[
1 + \varepsilon \geq \min \{2x_C, y_C\}
\]

One person coalition:

\[
1 + \varepsilon \geq 1 \text{ and } 2 - 4\varepsilon \geq 1
\]

\[
\Rightarrow 0 \leq \varepsilon \leq \frac{1}{4}
\]

Two people coalition:

\[
2 - 4\varepsilon \geq \min \left\{2 - \frac{1 + \varepsilon}{2}, 2 \cdot (2 - (1 + \varepsilon))\right\}
\]

\[
\Rightarrow 2 - 4\varepsilon \geq \min \left\{\frac{3}{2} - \frac{\varepsilon}{2}, 2 - 2\varepsilon\right\}
\]

\[
\Rightarrow 2 - 4\varepsilon \geq \frac{3}{2} - \frac{\varepsilon}{2} \left( \text{ since } \varepsilon \leq \frac{1}{4} \right)
\]

\[
\Rightarrow \frac{1}{2} \geq \frac{7}{2}\varepsilon
\]

\[
\Rightarrow \varepsilon \leq \frac{1}{7}
\]

Three people coalition:

It is Pareto optimal.

Therefore, any \( \varepsilon \in \left[0, \frac{1}{7}\right] \) implies the allocation is in the Core.

4 Question 4

(Comprehensive Exam August 2008 Q4)
4.1 Part a

Let $c^b_d$ represent the $a$-th period consumption of consumer $b$.

Core allocation requires:

1. Not blocked by $\{1\}, \{2\}, ..., \{N\}$:

$$\log (c^n_n) + \log (c^n_{n+1}) \geq \log (\theta) + \log (\theta') \text{ for } n > 0$$

$$\log (c^0_1) \geq \log (\theta')$$

$$c^N_{N+1} \leq \theta' \Rightarrow c^N_N < \theta \text{ are blocked by } \{N\}$$

$$\Rightarrow c^N_N \geq \theta \Rightarrow c^{N-1}_N \leq \theta' \Rightarrow c^{N-1}_{N-1} < \theta \text{ are blocked by } \{N-1\}$$

$$\Rightarrow c^{N-1}_{N-1} \geq \theta \Rightarrow c^{N-2}_{N-1} \leq \theta' \Rightarrow c^{N-2}_{N-2} < \theta \text{ are blocked by } \{N-2\}$$

$$\Rightarrow ...$$

$$\Rightarrow c^2_2 \geq \theta \Rightarrow c^1_2 \leq \theta' \Rightarrow c^1_1 < \theta \text{ are blocked by } \{1\}$$

Now,

$$c^0_1 < \theta' \text{ are blocked by } \{0\}$$

$$\Rightarrow c^1_1 \leq \theta \Rightarrow c^1_1 = \theta \Rightarrow c^1_2 < \theta' \text{ are blocked by } \{1\} \Rightarrow c^1_2 = \theta'$$

$$\Rightarrow c^2_2 \leq \theta \Rightarrow c^2_2 = \theta \Rightarrow c^2_3 < \theta' \text{ are blocked by } \{2\} \Rightarrow c^2_3 = \theta'$$

$$\Rightarrow ...$$

$$\Rightarrow c^N_N = \theta$$

Therefore, only consuming endowment is in the Core.

4.2 Part b

Walrasian equilibrium requires:
1. Consumers maximize utility:

\[ \max_{c_n, c_{n+1}} \log(c_n^n) + \log(c_{n+1}^{n+1}) \text{ such that } p_n c_n^n + p_{n+1} c_{n+1}^{n+1} = p_n \theta + p_{n+1} \theta' \]

\[ \max_{c_1^0} \log(c_1^0) \text{ such that } p_0 c_1^0 = p_0 \theta' \]

2. Markets clear:

\[ c_n^{n-1} + c_n^n = \theta + \theta' \]

\[ c_{N+1}^N = \theta' \]

Consider any sequence of prices \( \{p_n\}_{n=0}^N \):

Start from consumer 0, her budget constraint implies:

\[ c_0^0 = \theta' \]

Then market clearing condition of \( c_1 \) implies:

\[ c_1^1 = \theta + \theta' - c_1^0 = \theta + \theta' - \theta' = \theta \]

Then consumer 1’s budget constraint implies:

\[ c_2^1 = \theta' \]

Then market clearing condition of \( c_2 \) implies:

\[ c_2^2 = \theta + \theta' - c_2^1 = \theta + \theta' - \theta' = \theta \]

... \[ c_N^N = \theta \]

\[ c_{N+1}^N = \theta' \]

4.3 Part c

The argument in Part b is forward induction. Therefore it applies to \( N \to \infty \).
4.4 Part d

The argument in Part a is backward induction. Therefore it does not apply.

Consider a coalition of \{0, 1, 2, ...\}:

The allocation with:

\[ c_n^n = c_n^{n-1} = \frac{\theta + \theta'}{2} \text{ for } n \geq 0 \]

is preferred by all consumers since:

\[ \log \left( \frac{\theta + \theta'}{2} \right) > \log (\theta') \text{ for consumer 0} \]
\[ 2 \log \left( \frac{\theta + \theta'}{2} \right) > \log (\theta) + \log (\theta') \text{ for consumers } n > 0 \]

by concavity of log.

5 Question 5

(Comprehensive Exam June 2004 Q4)

5.1 Part a

Z is blocked by coalition \{A_1, A_2, B_1\}, need to check:

\[ u_{A1} (x_{A1}, y_{A1}) \geq u_{A1} (Z_{A1}^x, Z_{A1}^y) \]
\[ u_{A2} (x_{A2}, y_{A2}) \geq u_{A2} (Z_{A2}^x, Z_{A2}^y) \]
\[ u_{B1} (x_{B1}, y_{B1}) \geq u_{B1} (Z_{B1}^x, Z_{B1}^y) \]

with

\[ x_{A1} + x_{A2} + x_{B1} = 2\omega_A^x + \omega_B^x \]
\[ y_{A1} + y_{A2} + y_{B1} = 2\omega_A^y + \omega_B^y \]
Consider:

\[
\begin{align*}
  x_{A1} &= x_{A2} = \frac{1}{2} (\omega^x_A + Z^x_A) \\
  y_{A1} &= y_{A2} = \frac{1}{2} (\omega^y_A + Z^y_A) \\
  x_{B1} &= Z^x_B \\
  y_{B1} &= Z^y_B
\end{align*}
\]

where

\((x_{A1}, y_{A1})\) and \((x_{A2}, y_{A2})\) are located at the midpoint between \(\omega\) and \(Z\) (between \(Z\) and \(B\)), strictly preferred for \(A1\) and \(A2\).

\((x_{B1}, y_{B1})\) is located at \(Z\), indifferent for \(B1\).

and

\[
\begin{align*}
  x_{A1} + x_{A2} + x_{B1} \\
  &= \omega^x_A + Z^x_A + Z^x_B \\
  &= \omega^x_A + \omega^x_A + \omega^x_B \\
  y_{A1} + y_{A2} + y_{B1} \\
  &= \omega^y_A + Z^y_A + Z^y_B
\end{align*}
\]

5.2 Part b

Suppose there are \(N\) players of each type, then:

\(Z\) is blocked by coalition \(\{A_1, A_2, ..., A_N, B_1, B_2, ..., B_{N-1}\}\), need to check:

\[
\begin{align*}
  u_{A_i}(x_{A_i}, y_{A_i}) &\geq u_{A_i}(Z^x_A, Z^y_A) \text{ for } i \in \{1, 2, ..., N\} \\
  u_{B_j}(x_{B_j}, y_{B_j}) &\geq u_{B_j}(Z^x_B, Z^y_B) \text{ for } j \in \{1, 2, ..., N-1\}
\end{align*}
\]

with

\[
\begin{align*}
  \sum_{i=1}^{N} x_{A_i} + \sum_{j=1}^{N-1} x_{B_j} &= N\omega^x_A + (N - 1)\omega^x_B \\
  \sum_{i=1}^{N} y_{A_i} + \sum_{j=1}^{N-1} y_{B_j} &= N\omega^y_A + (N - 1)\omega^y_B
\end{align*}
\]
Consider:

\[ x_{Ai} = \frac{1}{N} \omega^x_A + \frac{N-1}{N} Z^x_A \text{ and} \]
\[ y_{Ai} = \frac{1}{N} \omega^y_A + \frac{N-1}{N} Z^y_A \text{ for } i \in \{1, 2, \ldots, N\} \]

\[ x_{Bj} = Z^x_B \text{ and} \]
\[ y_{Bj} = Z^y_B \text{ for } j \in \{1, 2, \ldots, N-1\} \]

where

\((x_{Ai}, y_{Aj})s\) are located at some point between \(\omega\) and \(Z\) for large \(N\) (between \(Z\) and \(B\)), strictly preferred for \(A_i\).

\((x_{Bj}, y_{Bj})\) is located at \(Z\), indifferent for \(B_j\).

and

\[
\sum_{i=1}^{N} x_{Ai} + \sum_{j=1}^{N-1} x_{Bj} \\
= \omega^x_A + (N-1)(Z^x_A + Z^x_B) \\
= \omega^x_A + (N-1)(\omega^x_A + \omega^x_B) \\
= N \omega^x_A + (N-1) \omega^x_B \\
\]

\[
\sum_{i=1}^{N} y_{Ai} + \sum_{j=1}^{N-1} y_{Bj} \\
= \omega^y_A + (N-1)(Z^y_A + Z^y_B) \\
= \omega^y_A + (N-1)(\omega^y_A + \omega^y_B) \\
= N \omega^y_A + (N-1) \omega^y_B \\
\]