ECO2020 Tutorial 4

Young Wu

December 7, 2016

1 Question 1

(Comprehensive Exam August 2004 Q4)

Arrow-Debreu equilibrium requires:

1. Consumers maximize utility:

$$\max_{x_{ns}} \sum_{s=1}^{S} \pi_s u_n \left(x_{ns}\right) \text{ such that } \sum_{s=1}^{S} p_s x_{ns} = \sum_{s=1}^{S} p_s \omega_{ns}$$

   (The budget constraint holds with equality since u_n is strictly increasing)

2. Markets clear:

$$\sum_{n=1}^{N} x_{ns} = \sum_{n=1}^{N} \omega_{ns}$$

$\Rightarrow$ Direction: $x^\star_{n\hat{s}} \geq x^\star_{n\bar{s}}$ implies $\sum_{n=1}^{N} \omega_{n\hat{s}} \geq \sum_{n=1}^{N} \omega_{n\bar{s}}$

$$\begin{aligned} & x^\star_{n\hat{s}} \geq x^\star_{n\bar{s}} \\ & \Rightarrow \sum_{=1}^{N} x^\star_{n\hat{s}} \geq \sum_{=1}^{N} x^\star_{n\bar{s}} \\ & \Rightarrow \sum_{n=1}^{N} \omega_{n\hat{s}} \geq \sum_{n=1}^{N} \omega_{n\bar{s}} \end{aligned}$$

$\Leftarrow$ Direction: $\sum_{n=1}^{N} \omega_{n\hat{s}} \geq \sum_{n=1}^{N} \omega_{n\bar{s}}$ implies $x^\star_{n\hat{s}} \geq x^\star_{n\bar{s}}$

Consider the Lagrangian of the consumer's problem:

$$\max_{x_{ns}} \sum_{s=1}^{S} \pi_s u_n(x_{ns}) - \lambda_n \left(\sum_{s=1}^{S} p_s x_{ns} - \sum_{s=1}^{S} p_s \omega_{ns} \right)$$
$$\Rightarrow \text{ FOC } : \pi_s u_n'(x_{ns}) = \lambda_n p_s$$
$$\Rightarrow u_n'(x_{ns}) = \lambda_n \frac{p_s}{\pi_s}$$
$$\Rightarrow \frac{u_n'(x_{n\hat{s}})}{u_n'(x_{n\bar{s}})} = \frac{p_{\hat{s}}\pi_{\bar{s}}}{p_{\bar{s}}\pi_{\hat{s}}}$$

Since $\frac{p_{\hat{s}}\pi_{\bar{s}}}{p_{\bar{s}}\pi_{\hat{s}}}$ is constant for all n, there are only two cases:

Case 1:

$$\frac{p_{\hat{s}}\pi_{\bar{s}}}{p_{\bar{s}}\pi_{\hat{s}}} > 1$$
$$\Rightarrow \frac{u_n'(x_{n\hat{s}})}{u_n'(x_{n\bar{s}})} > 1 \; \forall \; n = \{1, 2, ..., N\}$$
$$\Rightarrow x_{n\hat{s}} < x_{n\bar{s}} \; \forall \; n = \{1, 2, ..., N\}$$
$$\Rightarrow \sum_{n=1}^{N} x_{n\hat{s}} < \sum_{n=1}^{N} x_{n\bar{s}}$$
$$\Rightarrow \sum_{n=1}^{N} \omega_{n\hat{s}} < \sum_{n=1}^{N} \omega_{n\bar{s}}$$

Contraction.

Case 2:

$$\frac{p_{\hat{s}}\pi_{\bar{s}}}{p_{\bar{s}}\pi_{\hat{s}}} \leq 1$$
$$\Rightarrow \frac{u_n'(x_{n\hat{s}})}{u_n'(x_{n\bar{s}})} \leq 1 \; \forall \; n = \{1, 2, ..., N\}$$
$$\Rightarrow x_{n\hat{s}} \geq x_{n\bar{s}} \; \forall \; n = \{1, 2, ..., N\}$$

Last line is by strict concavity of u_n.

2 Question 2

(Comprehensive Exam August 2009 Q4)

2.1 Part a

Arrow-Debreu equilibrium requires:

1. Consumers maximize utility:

$$\max_{x_{ns}} \sum_{s=1}^{S} \pi_s u\left(x_s\right) \text{ such that } \sum_{s=1}^{S} p_s x_{ns} = \sum_{s=1}^{S} p_s$$

2. Markets clear:

$$\sum_{n=1}^{S} x_{ns} = p_n \text{ for } s \in \{1, 2, ..., S\}$$

2.2 Part b

From Question 1, FOC of the Lagrangian,

$$\frac{u'\left(x_{ns}\right)}{u'\left(x_{nt}\right)} = \frac{p_s \pi_t}{p_t \pi_s} \text{ for } n \in \{1, 2, ..., S\}$$

Suppose for a contradiciton, $\frac{p_s \pi_t}{p_t \pi_s} > 1$, then:

$$\frac{u'\left(x_{ns}\right)}{u'\left(x_{nt}\right)} > 1$$

$$\Rightarrow u'\left(x_{ns}\right) > u'\left(x_{nt}\right)$$

$$\Rightarrow x_{ns} > x_{nt}$$

$$\Rightarrow 1 = \sum_{n=1}^{S} x_{ns} > \sum_{n=1}^{S} x_{nt} = 1$$

Contradiction.

Similary, $\frac{p_s \pi_t}{p_t \pi_s} < 1$ leads to a contradiction.

Therefore, $\frac{p_s \pi_t}{p_t \pi_s} = 1$

$$\sum_{s=1}^{S} \pi_s = 1$$
$$\Rightarrow \sum_{s=1}^{S} \frac{p_s \pi_t}{p_t} = 1$$
$$\Rightarrow \frac{\pi_t}{p_t} \sum_{s=1}^{S} p_s = 1$$
$$\Rightarrow \pi_t = p_t \text{ if } \sum_{s=1}^{S} p_s = 1 \text{ normalized}$$

Also,

$$\frac{p_s \pi_t}{p_t \pi_s} = 1$$
$$\Rightarrow \frac{u'(x_{ns})}{u'(x_{nt})} = 1$$
$$\Rightarrow u'(x_{ns}) = u'(x_{nt})$$
$$\Rightarrow x_{ns} = x_{nt}$$

$$\sum_{s=1}^{S} p_s x_{ns} = p_n$$
$$\Rightarrow x_{nt} = \pi_n$$

3 Question 3

(Comprehensive Exam June 2014 Q4)

3.1 Part a

Arrow-Debreu equilibrium requires:

1. Consumers maximize utility:

$$\max_{x_a,y_a,z_a,w_a} x_a^{\frac{1}{6}} y_a^{\frac{1}{6}} z_a^{\frac{1}{3}} w_a^{\frac{1}{3}} \text{ such that } p_x x_a + p_y y_a + p_z z_a + p_w w_a = p_x + 2p_z$$
$$\max_{x_b,y_b,z_b,w_b} x_b^{\frac{1}{6}} y_b^{\frac{1}{6}} z_b^{\frac{1}{3}} w_b^{\frac{1}{3}} \text{ such that } p_x x_b + p_y y_b + p_z z_b + p_w w_b = p_y + 2p_z$$
$$\max_{x_c,y_c,z_c,w_c} x_c^{\frac{1}{6}} y_c^{\frac{1}{6}} z_c^{\frac{1}{3}} w_c^{\frac{1}{3}} \text{ such that } p_x x_c + p_y y_c + p_z z_c + p_w w_c = p_x + 2p_w$$
$$\max_{x_d,y_d,z_d,w_d} x_d^{\frac{1}{6}} y_d^{\frac{1}{6}} z_d^{\frac{1}{3}} w_d^{\frac{1}{3}} \text{ such that } p_x x_d + p_y y_d + p_z z_d + p_w w_d = p_y + 2p_w$$

2. Markets clear:

$$x_a + x_b + x_c + x_d = 1 + 1 = 2$$
$$y_a + y_b + y_c + y_d = 1 + 1 = 2$$
$$z_a + z_b + z_c + z_d = 2 + 2 = 4$$
$$w_a + w_b + w_c + w_d = 2 + 2 = 4$$

Normalize $p_w = 1$, then:

$$x_i = \frac{1}{6}\frac{I_i}{p_x} \text{ for } i \in \{a, b, c, d\}$$
$$y_i = \frac{1}{6}\frac{I_i}{p_y} \text{ for } i \in \{a, b, c, d\}$$
$$z_i = \frac{1}{3}\frac{I_i}{p_z} \text{ for } i \in \{a, b, c, d\}$$
$$w_i = \frac{1}{3}I_i \text{ for } i \in \{a, b, c, d\}$$

where I_i are incomes:

$$I_a = p_x + 2p_z$$
$$I_b = p_y + 2p_z$$
$$I_c = p_x + 2$$
$$I_d = p_y + 2$$

Market clearing conditions imply:

$$\frac{1}{6}\frac{I_a + I_b + I_c + I_d}{p_x} = 2$$
$$\frac{1}{6}\frac{I_a + I_b + I_c + I_d}{p_y} = 2$$
$$\frac{1}{6}\frac{I_a + I_b + I_c + I_d}{p_z} = 4$$
$$\frac{1}{6}(I_a + I_b + I_c + I_d) = 4$$
$$\Rightarrow I_a + I_b + I_c + I_d = 24$$
$$\Rightarrow p_x = p_y = p_z = 1$$
$$\Rightarrow I_a = I_b = I_c = I_d = 3$$
$$\Rightarrow x_a = x_b = x_c = x_d = y_a = y_b = y_c = y_d = \frac{1}{2}$$
$$\text{and } z_a = z_b = z_c = z_d = w_a = w_b = w_c = w_d = 1$$

3.2 Part b

Radner equilibrium requires:

1. Consumer maximizes utility:

   For each $i \in \{a, b, c, d\}$,

$$\max_{x_i, y_i, z_i, w_i} x_i^{\frac{1}{6}} y_i^{\frac{1}{6}} + z_i^{\frac{1}{3}} + w_i^{\frac{1}{3}} \text{ such that}$$
$$q \cdot \zeta_i = 0$$
$$p_x x_i \leq p_x \omega_i^x + p_x \zeta_i \cdot r_x$$
$$p_y y_i \leq p_y \omega_i^y + p_y \zeta_i \cdot r_y$$
$$p_z z_i \leq p_z \omega_i^z + p_z \zeta_i \cdot r_w$$
$$p_w w_i \leq p_w \omega_i^w + p_w \zeta_i \cdot r_z$$

   Here, ζ_i is the portfolio (instead of z_i) to avoid confusion of notations.

2. Markets clear:

$$\sum_{i\in\{a,b,c,d\}} x_i = \sum_{i\in\{a,b,c,d\}} \omega_i^x$$
$$\sum_{i\in\{a,b,c,d\}} y_i = \sum_{i\in\{a,b,c,d\}} \omega_i^y$$
$$\sum_{i\in\{a,b,c,d\}} z_i = \sum_{i\in\{a,b,c,d\}} \omega_i^z$$
$$\sum_{i\in\{a,b,c,d\}} w_i = \sum_{i\in\{a,b,c,d\}} \omega_i^w$$
$$\sum_{i\in\{a,b,c,d\}} \zeta_i = 0$$

If the spanning conditions hold, the set of WE (ADE) allocations is the same as the set of RE allocations.

Therefore, at most four financial assets are needed:

$$r^1 = (1,0,0,0)$$
$$r^2 = (0,1,0,0)$$
$$r^3 = (0,0,1,0)$$
$$r^4 = (0,0,0,1)$$

3.3 Part c

Case 1 : 1 asset r^1, then:

$$\frac{1}{2} = 1 + r_x^1 z_a^1$$
$$\frac{1}{2} = 0 + r_y^1 z_a^1$$
$$1 = 2 + r_z^1 z_a^1$$
$$1 = 0 + r_w^1 z_a^1$$
$$\frac{1}{2} = 0 + r_x^1 z_b^1$$
$$\frac{1}{2} = 1 + r_y^1 z_b^1$$
$$1 = 2 + r_z^1 z_b^1$$
$$1 = 0 + r_w^1 z_b^1$$

which implies:

$$-2r_x^1 = 2r_y^1 = -r_y^1 = r_w^1$$
$$2r_x^1 = -2r_y^1 = -r_y^1 = r_w^1$$

$r^1 = (0,0,0,0)$ Not possible.

Case 2 : 2 assets r^1, r^2, then:

$$\frac{1}{2} = 1 + r_x^1 z_a^1 + r_x^2 z_a^2$$
$$\frac{1}{2} = 0 + r_y^1 z_a^1 + r_y^2 z_a^2$$
$$1 = 2 + r_z^1 z_a^1 + r_z^2 z_a^2$$
$$1 = 0 + r_w^1 z_a^1 + r_w^2 z_a^2$$
$$\frac{1}{2} = 0 + r_x^1 z_b^1 + r_x^2 z_b^2$$
$$\frac{1}{2} = 1 + r_y^1 z_b^1 + r_y^2 z_b^2$$
$$1 = 2 + r_z^1 z_b^1 + r_z^2 z_b^2$$
$$1 = 0 + r_w^1 z_b^1 + r_w^2 z_b^2$$
$$\frac{1}{2} = 1 + r_x^1 z_c^1 + r_x^2 z_c^2$$
$$\frac{1}{2} = 0 + r_y^1 z_c^1 + r_y^2 z_c^2$$
$$1 = 0 + r_z^1 z_c^1 + r_z^2 z_c^2$$
$$1 = 2 + r_w^1 z_c^1 + r_w^2 z_c^2$$
$$\frac{1}{2} = 0 + r_x^1 z_d^1 + r_x^2 z_d^2$$
$$\frac{1}{2} = 1 + r_y^1 z_d^1 + r_y^2 z_d^2$$
$$1 = 0 + r_z^1 z_d^1 + r_z^2 z_d^2$$
$$1 = 2 + r_w^1 z_d^1 + r_w^2 z_d^2$$

which implies the following is one solution:

$$
\begin{aligned}
r^1 &= (1, -1, 0, 0) \\
r^2 &= (0, 0, 2, -2) \\
z_a &= \left(-\frac{1}{2}, -\frac{1}{2}\right) \\
z_b &= \left(\frac{1}{2}, -\frac{1}{2}\right) \\
z_c &= \left(-\frac{1}{2}, \frac{1}{2}\right) \\
z_d &= \left(\frac{1}{2}, \frac{1}{2}\right)
\end{aligned}
$$

4 Question 4

4.1 Part a

Walrasian equilibrium requires:

1. Consumers maximize utility:

$$A_n \in \arg\max_{S_n} \beta_i \frac{1-\beta_i^{|S_n|}}{1-\beta_i} - \sum_{m \in S_n} p(m)$$

2. Markets clear:

$$
\begin{aligned}
&\cup_{n=1}^{N} A_n = \Omega \text{ with } A_i \cap A_j = \emptyset \; \forall \; i \neq j \\
&p(m) = 0 \text{ for } m \in A_0
\end{aligned}
$$

4.2 Part b

$\lvert S_n\rvert$	1	2	3	4
u_1	$\frac{1}{2}\frac{1-\left(\frac{1}{2}\right)^1}{1-\frac{1}{2}}$	$\frac{1}{2}\frac{1-\left(\frac{1}{2}\right)^2}{1-\frac{1}{2}}$	$\frac{1}{2}\frac{1-\left(\frac{1}{2}\right)^3}{1-\frac{1}{2}}$	$\frac{1}{2}\frac{1-\left(\frac{1}{2}\right)^4}{1-\frac{1}{2}}$
u_2	$\frac{1}{5}\frac{1-\left(\frac{1}{5}\right)^1}{1-\frac{1}{5}}$	$\frac{1}{5}\frac{1-\left(\frac{1}{5}\right)^2}{1-\frac{1}{5}}$	$\frac{1}{5}\frac{1-\left(\frac{1}{5}\right)^3}{1-\frac{1}{5}}$	$\frac{1}{5}\frac{1-\left(\frac{1}{5}\right)^4}{1-\frac{1}{5}}$

Also compute $\Delta u_i = u_i(|S_n|) - u_i(|S_n| - 1)$:

$\|S_n\|$	1	2	3	4
u_1	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{15}{16}$
Δu_1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
u_2	$\frac{1}{5}$	$\frac{6}{25}$	$\frac{31}{125}$	$\frac{156}{625}$
Δu_2	$\frac{1}{5}$	$\frac{1}{25}$	$\frac{1}{125}$	$\frac{1}{625}$

Any $p \in \left(\frac{1}{16}, \frac{1}{8}\right]$ with any partition $\{A_0, A_1, A_2\}$ such that:

$$A_0 = \emptyset$$

$$|A_1| = 3$$

$$|A_2| = 1$$

results in a WE.

4.3 Part c

Any price in $\left[0, \frac{1}{16}\right]$, $|D_1| = 4$ and $|D_2| > 1$ infeasible.

Any price in $\left(\frac{1}{8}, \infty\right)$, $|D_1| < 3$ and $|D_2| \leq 1$, there will be unsold good (see Part d).

Therefore only $p \in \left(\frac{1}{16}, \frac{1}{8}\right]$ are WE prices.

4.4 Part d

If j is not demanded,

$$p(j) = 0$$

Then

$$p(i) = p(j) = 0 \text{ for } i \in \{1, 2, ..., J\}$$

since all goods are identical.

4.5 Part e

The example with $J = 1, N = 2, \beta_1 = \beta_2 = \frac{1}{2}$ given in tutorial is NOT a counterexaple:

It has WE with $p = \frac{1}{2}, A_0 = A_1 = \emptyset, A_2 = 1$.

To prove there is a WE for any J, N, β:

Consider the marginal utility defined above:

$$\begin{aligned}
\Delta u_i(k) &= \beta_i \frac{1-\beta_i^k}{1-\beta_i} - \beta_i \frac{1-\beta_i^{k-1}}{1-\beta_i} \\
&= \beta_i \frac{\beta_i^{k-1} - \beta_i^k}{1-\beta_i} \\
&= \beta_i \left(\beta_i^{k-1}\right) \\
&= \beta_i^k
\end{aligned}$$

is decreasing in k since $\beta_i \in (0,1)$.

It implies that if $\Delta u_i(k^\star) \geq p$, then $\Delta u_i(k) \geq p$ for all $k < k^\star$.

Now order the $\Delta u_i(k)$ in decreasing order:

$$\left\{\Delta^{(1)}u, \Delta^{(2)}u, ..., \Delta^{(N \cdot J)}u\right\}$$

Take the first J elements:

$$S_J = \left\{\Delta^{(1)}u, \Delta^{(2)}u, ..., \Delta^{(J)}u\right\}$$

Note that since Δu is decreasing in k, if $\Delta u_i(k^\star)$ is in the set, then $\Delta u_i(k)$ is too for all $k < k^\star$.

Consider the following allocations:

$$\begin{aligned}
&\text{any partition } \{A_0, A_1, ..., A_N\} \text{ such that } A_0 = \emptyset \text{ and } |A_i| = k_i \\
&\qquad\qquad \text{where } k_i = \max_k \{k : \Delta u_i(k) \in S_J\} \\
&\text{with } p \in \left[\Delta^{(J)}u, \Delta^{(J+1)}u\right)
\end{aligned}$$

Note that $\sum_{i=1}^{N} k_i = J$ and $|D_i| = k_i$ by construction. Therefore the above is a WE.

5 Question 5

(Comprehensive Exam June 2008 Q4)

5.1 Part a

Walrasian equilibrium requires:

1. Consumers maximize utility:

$$A_n \in \arg\max_{S_n} \max_{m \in S_n} u_n \theta_m - \sum_{m \in S_n} p(m)$$

2. Markets clear:

$$\cup_{n=1}^{N} = A_n = \Omega \text{ with } A_i \cap A_j = \emptyset \; \forall \, i \neq j$$
$$p(m) = 0 \text{ for } m \in A_0$$

Assume otherwise and $p(m) \leq p(m')$ for $m > m'$:

$$u_n \theta_m - p(m) > u_n \theta_{m'} - p(m')$$

Then m' is not in the demand set of any consumer:

$$m' \in A_0$$
$$\Rightarrow p(m') = 0$$
$$\Rightarrow p(m) \leq p(m') = 0$$

Contradiction, since then m will be overdemanded by every consumer.

5.2 Part b

If $p(1) < u_{N-M}\theta_1$ then:

$$\begin{cases} u_{N-M}\theta_1 - p(1) > 0 \\ u_{N-M+1}\theta_1 - p(1) > u_{N-M}\theta_1 - p(1) > 0 \\ \ldots \\ u_N\theta_1 - p(1) > u_{N-M}\theta_1 - p(1) > 0 \end{cases}$$

There are $M+1$ consumers demanding some subset of $\{1, 2, ..., M\}$, meaning it is overdemanded.

5.3 Part c

Assume $p(1) = u_{N-M}\theta_1$,

If $p(2) < u_{N-M+1}(\theta_2 - \theta_1) + p(1)$, then:

$$\begin{cases} u_{N-M+1}\theta_2 - p(2) > u_{N-M+1}\theta_1 - p(1) > 0 \\ u_{N-M+2}\theta_2 - p(2) > u_{N-M+1}\theta_1 - p(1) > 0 \\ \ldots \\ u_N\theta_2 - p(2) > u_{N-M+1}\theta_1 - p(1) > 0 \end{cases}$$

There are M consumers demanding some subset of $\{2, 3, ..., M\}$, meaning it is overdemanded.

5.4 Part d

Define the prices as before recursively:

$$\begin{aligned} p(1) &= u_{N-M}\theta_1 \\ p(2) &= u_{N-M+1}(\theta_2 - \theta_1) + p(1) \\ p(3) &= u_{N-M+2}(\theta_3 - \theta_2) + p(2) \\ \ldots &== \\ p(M) &= u_{N-1}(\theta_M - \theta_{M-1}) + p(M-1) \end{aligned}$$

Use strong induction and $p(1), p(2) ..., p(m-1)$ are specified as above,

If $p(m) < u_{N-M+m-1}(\theta_m - \theta_{m-1}) + p(m-1)$, then:

$$\begin{cases} u_{N-M+m-1}\theta_m - p(m) > u_{N-M+m-1}\theta_{m-1} - p(m-1) > 0 \\ u_{N-M+m}\theta_m - p(m) > u_{N-M+m-1}\theta_{m-1} - p(m-1) > 0 \\ \ldots \\ u_N\theta_m - p(m) > u_{N-M+m-1}\theta_{m-1} - p(m-1) > 0 \end{cases}$$

There are $M - m + 2$ consumers demanding some subset of $\{m, m+1, ..., M\}$, meaning it is overdemanded.

With these prices, for consumer n:

$$u_n\theta_{n-N+M} - p(n-N+M) = u_n\theta_{n-N+M+1} - p(n-N+M+1)$$

$$u_{n+1}\theta_{n-N+M+1} - p(n-N+M+1) = u_{n+1}\theta_{n-N+M+2} - p(n-N+M+2)$$

$$\Rightarrow u_n\theta_{n-N+M+1} - p(n-N+M+1) > u_n\theta_{n-N+M+2} - p(n-N+M+2)$$

$$u_{n-1}\theta_{n-N+M-1} - p(n-N+M+1) = u_{n-1}\theta_{n-N+M} - p(n-N+M)$$

$$\Rightarrow u_n\theta_{n-N+M-1} - p(n-N+M+1) < u_n\theta_{n-N+M} - p(n-N+M)$$

Therefore, the demand set for consumr n is:

$$\begin{cases} 0 & \text{if } n < N-M \\ \{n-N+M, n-N+M-1\} & \text{if } n > N-M \end{cases}$$

The allocation could be:

$$A_0 = A_1 = ... = A_{N-M} = \emptyset$$

$$A_{N-M+1} = \{1\}$$

$$A_{N-M+2} = \{2\}$$

$$...$$

$$A_N = \{M\}$$