CS368 MATLAB Programming

Lecture 12

Young Wu

Based on lecture slides by Michael O'Neill and Beck Hasti

April 20, 2022

Debugger, Strange Bug 1

Quiz

1. $x = 0;\ y = 0.3;\ z = 0;$
2. $for\ \ t = 1{:}5$
3. $\quad x = x + 0.1;$
4. $\quad if\ \ x == y$ while
5. $\qquad z = x;$
6. $\quad end$
7. end
8. z

- $\boxed{A : 0}$
- $B : 0.3$ ← should be
- $C : 0.5$

$$|x - y| < \underbrace{0.00001}_{\varepsilon}$$

$$abs(x - y) < 0.00001$$

Debugger, Strange Bug 2

Quiz

1. $q = 1;$
2. $for\ i = 1{:}2$
3. $\quad q = q * 2i$
4. end
5. q

- $A: -4$
- $B: 1$
- $C: 8$

MATLAB don't use i, j

$2\sqrt{-1}$

$2 * i$

$2 * \cancel{x}$

$\sqrt{-1}$

$\underbrace{1 \cdot 2i \cdot 2i}$

$4 \cancel{\sqrt{-1} \cdot \sqrt{-1}}$

-1

$i=1 \qquad i=2$

$\underline{1 \cdot 2 \cdot 2 \cdot 2} = 8$

← should be

Grades

Admin

- $P4$ to $P6$ code check tonight or tomorrow.
- Check your Canvas grades to make sure they are correct.
- Credit if grade $\geqslant 75$, no curve.
- If your current grade on Canvas in less than 50, email me.

Numerical Differentiation

Math

x^2

$2x$

- Derivatives can be approximated using Newton's formula,

$$\frac{df}{dx} \approx \frac{f(x+h) - f(x)}{h},$$

$\cancel{\lim_{h \to 0}}$

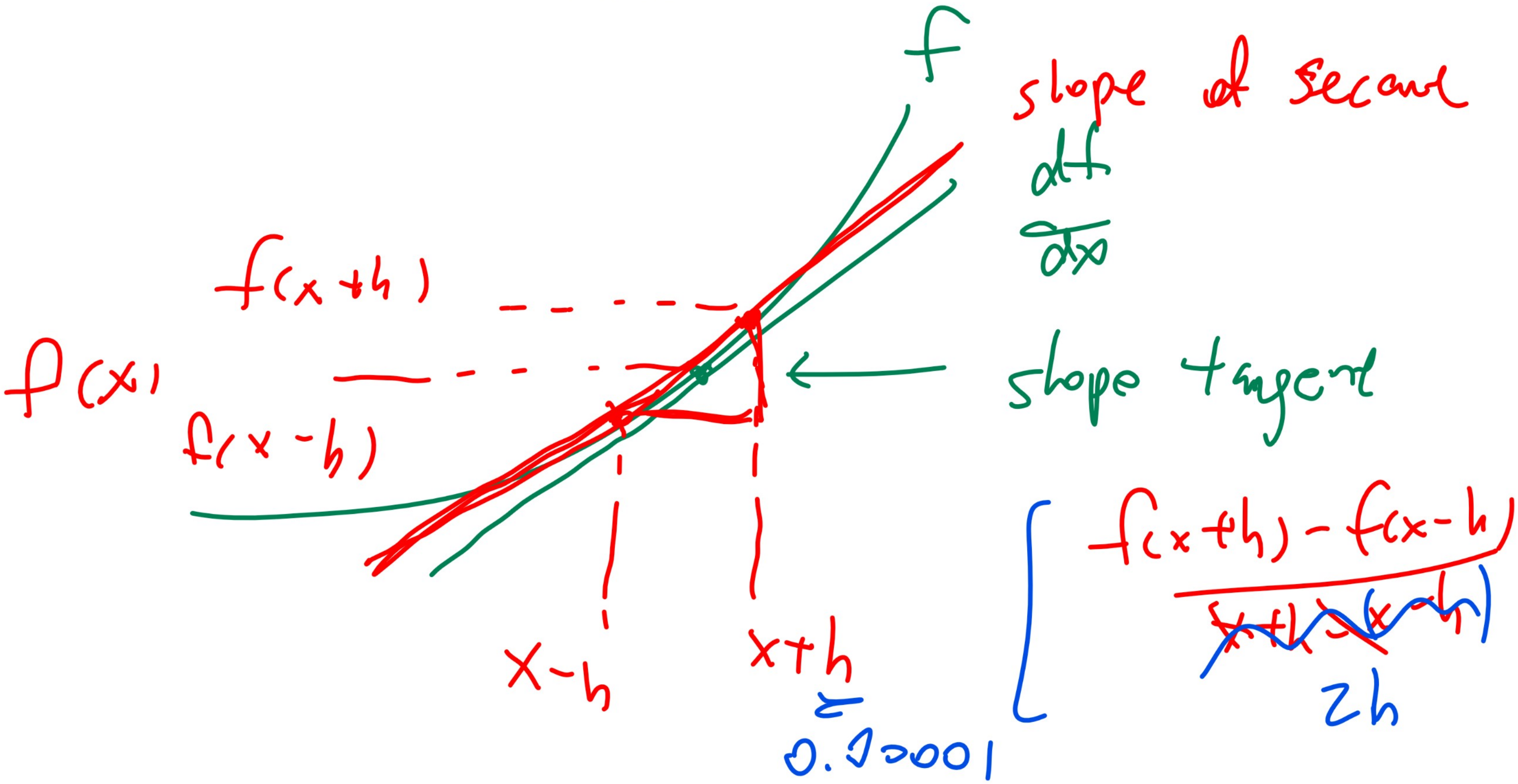
for some small h close to 0.

- The derivative of f at x is usually approximated by,

$$\frac{df}{dx} \approx \frac{f(x+h) - f(x-h)}{2h}.$$

Numerical Differentiation Diagram

Math



Finite Differences

Math

- There are more accurate approximations using finite differences, for example,

$$\frac{df}{dx} \approx \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h},$$

for some small h close to 0.

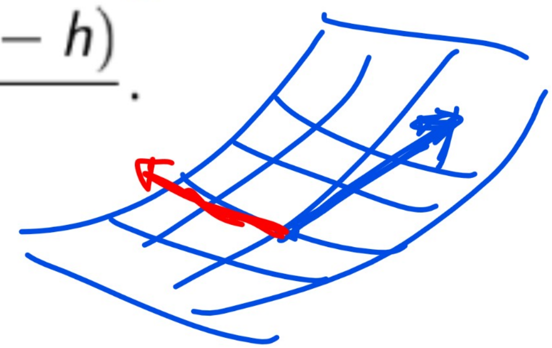
Partial Derivatives

Math

- Partial derivatives of a multivariate function are derivatives with respect to one of the variables holding the other variables constant.

$$\frac{\partial f(x,y)}{\partial x} \approx \frac{f(x+h,y) - f(x-h,y)}{2h}.$$

$$\frac{\partial f(x,y)}{\partial y} \approx \frac{f(x,y+h) - f(x,y-h)}{2h}.$$



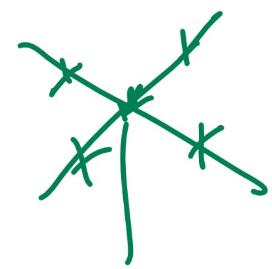
Partial Derivatives Too

Math

- Higher order derivatives can be approximated in a similar way,

$$\frac{\partial^2 f(x,y)}{\partial x^2} \approx \frac{f(x+h,y) - 2f(x,y) + f(x-h,y)}{h^2}.$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} \approx \frac{f(x,y+h) - 2f(x,y) + f(x,y-h)}{h^2}.$$



x,y

and,

$$\frac{\partial f(x,y)}{\partial x \partial y} \approx \frac{f(x+h,y+h) + f(x-h,y-h)}{4h^2} - \frac{f(x+h,y-h) + f(x-h,y+h)}{4h^2}.$$

Gradient

Math

- The gradient of a multivariate function is the vector of (partial) derivatives, one for each variable.

$$\nabla f(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}.$$

Derivative

Quiz

Q3

- $\left.\frac{d}{dx}x^2\right|_{x=2} = \mathbf{4}$ function handle $f = x.\hat{}2$

1. $f = @(x)(x\ .\hat{}\ 2);\ z = 2;\ h = 0.0001;$

- B : $(f(z+h) - f(z-h))\ /\ (2 * h)$
- C : $(f(z+h)\cancel{+}f(z-h))\ /\ (2 * h)$

Gradient

Quiz

Q4

- $$\frac{\partial}{\partial x_2}\left(x_1^2 + x_2^2 + x_3^2\right)\Bigg|_{x=\begin{bmatrix}1\\2\\3\end{bmatrix}} = \mathbf{4}$$

1. $f = @(x)(sum(x\ .\hat{}2));\ z = [1\ 2\ \ 3];\ \ h = 0.0001;$

- $B : (f(z + h) - f(z - h))\ /\ (2 * h)$
- $\boxed{C :}\ (f(z + [0\ h\ \ 0]) - f(z - [0\ h\ \ 0]))\ /\ (2 * h)$

$$f\left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ h \\ 0 \end{bmatrix}\right)$$

put 0 for Q13
0 for P7

Derivative as a Function

Code

- It is possible to compute the numerical derivative as a function ($\boxed{\text{a MATLAB function}}$, not a closed form expression of the function).

```
function d = df(f)
    h = 0.0001;
    d = @(x)((f(x + h) - f(x - h)) / (2 * h));
end
```

$f = x^2$

x^2

$df(f)$

$\boxed{2x}$

$d(1)$ $d(2)$

Automatic Differentiation

Code

- *diff (y)* computes the difference between consecutive elements of y and returns $(y_2 - y_1, y_3 - y_2, \ldots, y_n - y_{n-1})$ and can be used to approximate the derivative of the discretized function $y = f(x)$.
- *diff (f)* computes the derivative of f and returns a function. It requires MATLAB's Symbolic Math Toolbox.

$\text{diff}(x^2) \rightarrow 2x$

Numerical Integration or Quadrature

Math

- Definite integrals can be approximated using finite Riemann sums.

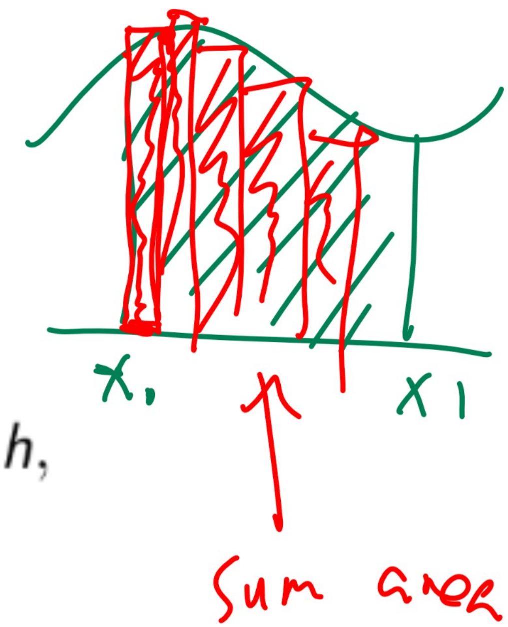
$$\int_a^b x^2 = \frac{b^3}{3} - \frac{a^3}{3}$$

- If the right Riemann sum is used, then,

$$\int_{x_0}^{x_1} f(x)\, dx \approx \sum_{i=1}^{n} f(x_0 + i \cdot h)\, h,$$

$\approx h$

for $h = \dfrac{x_1 - x_0}{n}$, for some large n.



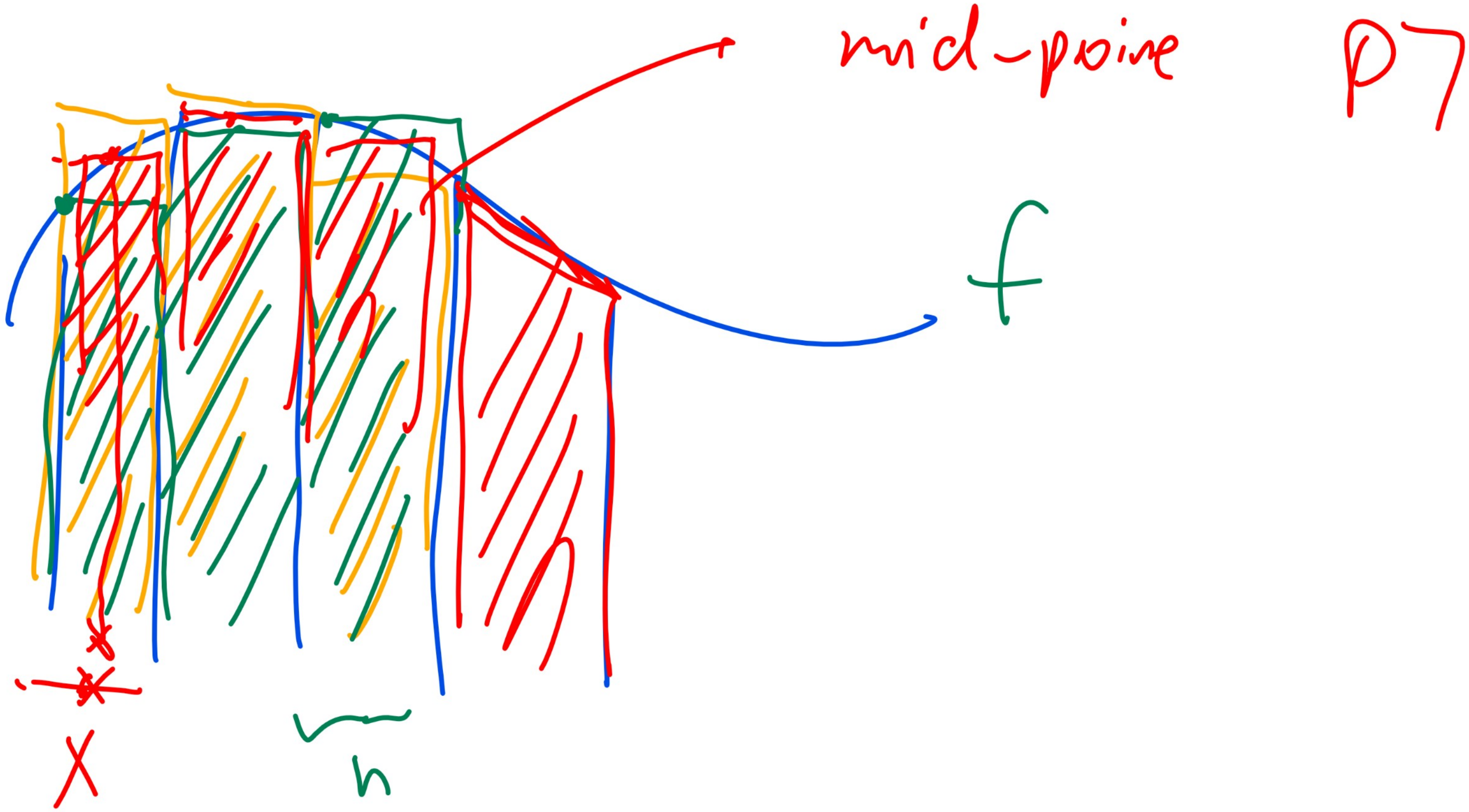
- If the left Riemann sum is used, then,

$$\int_{x_0}^{x_1} f(x)\, dx \approx \sum_{i=1}^{n} f(x_0 + (i-1) \cdot h)\, h,$$

for $h = \dfrac{x_1 - x_0}{n}$, for some large n.

Numerical Integration Diagram

Math



Midpoint Rule

- If the midpoint rule is used, then,

$$\int_{x_0}^{x_1} f(x)\,dx \approx \sum_{i=1}^{n} f\left(x_0 + (i - 0.5)\cdot h\right) h,$$

for $h = \dfrac{x_1 - x_0}{n}$, for some large n.

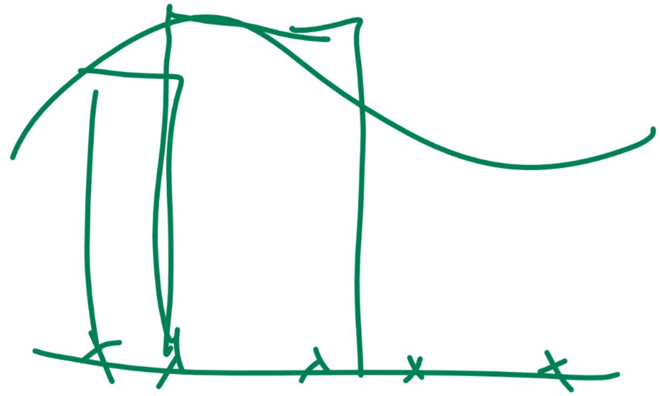
- If the trapezoidal rule is used, then,

$$\int_{x_0}^{x_1} f(x)\,dx \approx \sum_{i=1}^{n} \frac{1}{2}\left(f(x_0 + i\cdot h) + f(x_0 + (i-1)\cdot h)\right) h,$$

for $h = \dfrac{x_1 - x_0}{n}$, for some large n.

Adaptive and Monte Carlo

Math



- The points at which the function is evaluated can be chosen adaptively or randomly.
- For example, smaller h can be used when $f'(x)$ is large and larger h can be used when $f'(x)$ is small.

Double Integral

Math

- Finite Riemann sums can be used to approximate multiple integrals too.

$$\int_{x_0}^{x_1} \int_{y_0}^{y_1} f(x, y)\, dxdy$$

$$\approx \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_0 + (i - 0.5) \cdot h, y_0 + (j - 0.5) \cdot k) \cdot hk,$$

for $h = \frac{x_1 - x_0}{n}, k = \frac{y_1 - y_0}{m}$, for some large n, m.

- If $n \times m$ is too large, sometimes, Monte Carlo methods are used instead.

Indefinite Integral as a Function

Code

- It is possible to compute the numerical integral as a function (a MATLAB function, not a closed form expression of the function). Instead of having the $+C$ at the end, an arbitrary constant is used.

```
1 function d = df(f)
2   h = 0.01;
3   d = @(x)(sum(f(h:h:x)) * h);
4 end
```

$F(t) = \int_0^t f(x)\,dx$

$+C$

- This approximation uses the right Riemann sum and it does not work on improper integrals.

Integration

Code

- *integral(f, x0, x1)* finds the numerical definite integral $\int_{x_0}^{x_1} f(x)\,dx$.
- *int(f)* finds the indefinite integral of f and returns a function. It requires MATLAB's Symbolic Math Toolbox.

$$\int x^2 dx \Rightarrow \frac{x^3}{3}$$

Integral

Quiz

Q5

- $\int_0^1 x^2 =$ **0.3333**

1. *f = @(x)(x .^ 2);*

- C : *sum(~~f(0:0.01:1)~~) * 0.01*
- D : *sum(f(0.005:0.01:0.995)) * 0.01*

midpoint.

midpt of first partition.

Q15

course evaluation

Improper Integral

Quiz

- $\int_1^\infty \frac{1}{x^2} dx = \mathbf{1}$

1. $f = @(x)\ (x\ .\hat{}\ -2);$

- $C : x = 1.5{:}1{:}999.5;\ \ sum(f(x)) * 1$
- $D : x = 0.005{:}0.01{:}0.995;$
- $sum(f(1\ ./\ (1 - x))\ .*\ (1 - x)\ .\hat{}\ -2) * 0.01$

Double Integral

Quiz

- $\int_0^1 \int_0^1 xydxdy = \mathbf{0.25}$

1. $f = @(x, y)(x \ .* \ y);$

- $C : x = 0.05{:}0.01{:}0.995; \ \ y = 0.05{:}0.01{:}0.995;$
- $D : [x, \ y] = meshgrid(0.05{:}0.01{:}0.995, \ \ 0.05{:}0.01{:}0.995);$

3. $sum(sum(f(x, y))) * 0.01 * 0.01$

Blank Slide