CS368 MATLAB Programming

Lecture 6

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Based on lecture slides by Michael O'Neill and Beck Hasti

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Guess Two-Thirds of the Average Game

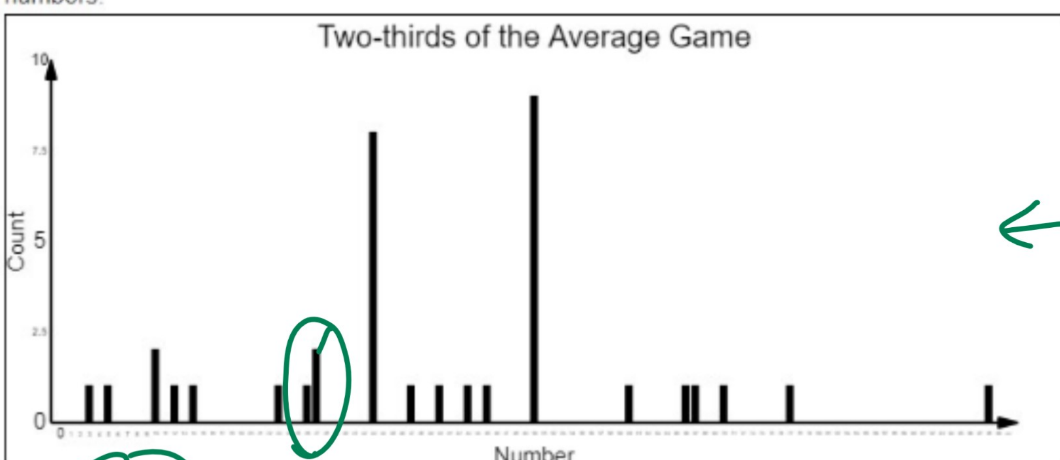
Quiz

- Enter an integer between 0 and 100 (including 0 and 100) that is the closest to $\frac{2}{3}$ of the average of everyone's integer.

Q1

Two-Thirds of the Average Game

• Round 1: Everyone selects a number between 0 and 100 and tries to be close to two-thirds of the average of all numbers.



last week

Average: 39.84

Two-thirds of the average: 26.67

A Simple Linear System

Quiz

- What is the height of the table?

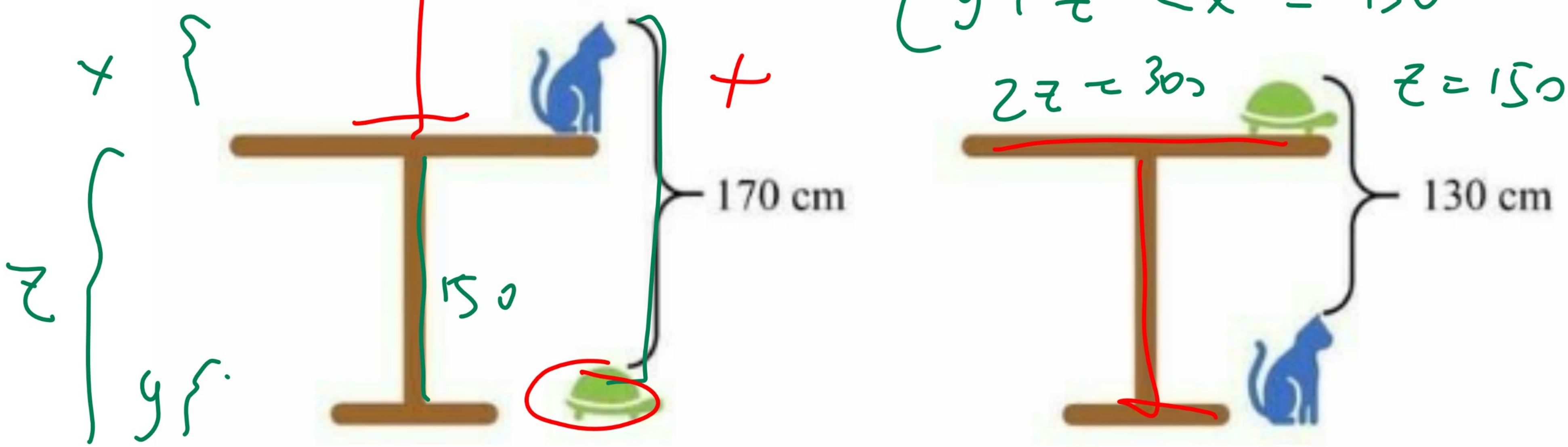
130

Q2

$$\begin{cases} x + z - y = 170 \\ y + z - x = 130 \end{cases}$$

$2z = 300$

$z = 150$



System of Two Equations

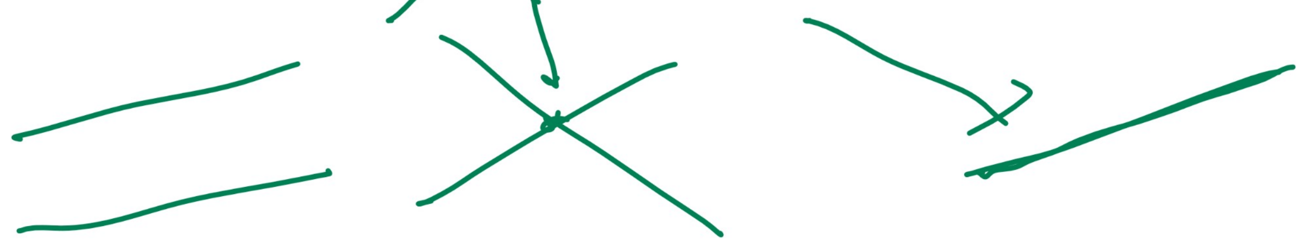
Math

- $\begin{cases} m_{11}x_1 + m_{12}x_2 = b_1 \\ m_{21}x_1 + m_{22}x_2 = b_2 \end{cases}$ is a system of two equations and two unknowns x_1 and x_2.
- The system can be written in matrix form

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

$mx = b$

- The system may have $0, 1$ or infinite number of solutions.



Two Equations Example

Math

- $\begin{array}{cc} \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \\ \begin{matrix} 1 & \;\; 2 \end{matrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 4 \\ 5 \end{bmatrix} \\ 3 \end{array}$ has no solution.
- $\begin{array}{c} \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \\ \begin{matrix} 1 & \;\; 2 \end{matrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 4 \\ 12 \end{bmatrix} \\ 4 \end{array}$ has infinite number of solutions, $x = \begin{bmatrix} 4 - 2t \\ t \end{bmatrix}, t \in \mathbb{R}$.
- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$ has a unique solution $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Two Equations General Solution

Math

$\frac{m_{11}}{m_{21}} \neq \frac{m_{12}}{m_{22}}$

$\underbrace{m_{11} m_{22} - m_{12} m_{21}}_{\text{det}} = 0$

- In general, for $\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, if $d = m_{11}m_{22} - m_{12}m_{21} = 0$, then there are either no solution or infinite number of solutions; otherwise, there is a unique solution $x = \frac{1}{d} \begin{bmatrix} m_{22}b_1 - m_{12}b_2 \\ m_{11}b_2 - m_{21}b_1 \end{bmatrix}$.

Solving Two Equations

Code

b/m   m\b

$x = \frac{b}{m}$

- *[m11 m12; m21 m22] \ [b1; b2]* or *[b1; b2] / [m11 m12; m21 m22]* solves the system
$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \text{ for } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

$mx = b$

- The MATLAB command will always output a solution, usually close to infinity if there are none; and only output one if there are infinite number of them.
- There will be a warning about the matrix being singular.

System of Equations

Math

- $$\begin{cases} m_{11}x_1 + m_{12}x_2 + ... + m_{1n}x_n = b_1 \\ m_{21}x_1 + m_{22}x_2 + ... + m_{2n}x_n = b_2 \\ ... \\ m_{k1}x_1 + m_{k2}x_2 + ... + m_{kn}x_n = b_k \end{cases}$$ is a system of k equations and n unknowns $x_1, x_2, ..., x_n$.
- The system can be written in matrix form
$$\begin{bmatrix} m_{11} & m_{12} & ... & m_{1n} \\ m_{21} & m_{22} & ... & m_{2n} \\ ... & ... & ... & ... \\ m_{k1} & m_{k2} & ... & m_{kn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ ... \\ b_k \end{bmatrix}$$, or $Mx = b$.

Solving a System

Math

- Let $Mx = b$ be a system of linear equations.
- x can be solved by hand using Gaussian elimination.
- It is faster to solve for x using LU decomposition.

LU Decomposition

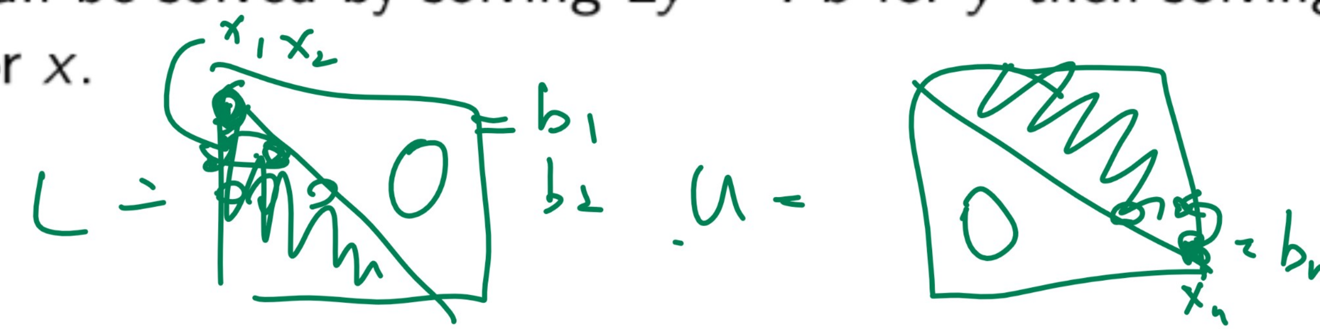
Math

- Every matrix M can be written in the form $PM = LU$.

1. P is a permutation matrix (each row and each column contain one 1 and all other entries are 0s).
2. L is a lower triangular matrix (all entries above the diagonal are 0s).
3. U is a upper triangular matrix (all entries below the diagonal are 0s).

$$\overset{LU}{\cancel{PM}}x = Pb \qquad \boxed{Ly = Pb \quad Ux = y}$$

- $Mx = b$ can be solved by solving $Ly = Pb$ for y then solving $Ux = y$ for x.



Two by Two LU Decomposition

Math

- $M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ can be written as
$$LU = \begin{bmatrix} 1 & 0 \\ \frac{m_{21}}{m_{11}} & \frac{1}{m_{11}} \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ 0 & m_{11}m_{22} - m_{12}m_{21} \end{bmatrix}.$$
- Here, $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the permutation matrix assuming $m_{11} \neq 0$, and if $m_{11} = 0, P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ can be used instead.
- This makes solving $Ly = b$ and $Ux = y$ very fast since only forward and backward substitutions are required.

Solving

Code

$$Mx = b_1$$
$$Mx = b_2$$
$$\vdots$$

- $M \setminus b$ or $b \ / \ M$ solves $Mx = b$ for x.
- $[L, \ U, \ P] = lu(M)$ computes the LU decomposition and $y = L \setminus (P * b); \ x = U \setminus y$ also solves $Mx = b$ for x.
- In the case $Mx = b$ needs to be solved repeatedly for the same M but different b's, it is faster to find the LU decomposition once and use L, U, P on different b's.
- $d = decomposition(M, \text{'lu'}); \ d \setminus b$ uses the same LU decomposition approach without the need to remember how to solve for x given the decomposition.

$d \setminus b$
$d \setminus b_1$
$d \setminus b_2$
$\vdots$

Linear System, Simple

Quiz

Q3

- (Solve $x_1 - x_2 = 0$ and $x_1 + x_2 = 2$.)
- **1 1**
- A : *[1 −1; 1 1] \ [0; 2]*
- B : *[1 1; −1 1] \ [0; 2]*

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Linear System, Component

Quiz

Q4

- (In a factor input matrix M, row i column j represents the amount of material i required in the production of product j. In an input vector b, row i represents the amount material i available. Given M, b, compute the number of products that can be produced.)

$$1x_1 + 2x_2 + 3x_3 = 12$$

m_1 m_2 m_3 m_1

- **1 1 1** ?

1. $M = [1\ 2\ 3;\ 4\ 5\ 6;\ 7\ 8\ 10];\ b = [12;\ 15;\ 19];$

product

- A : $M \setminus b$
- B : $M' \setminus b$

transpose.

material →

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 12 \\ 15 \\ 19 \end{bmatrix}$$

Linear System, Temperature

Quiz

Q5

- (In $\begin{bmatrix} \boxed{6} & \boxed{T_1} & \boxed{T_2} & - \\ - & \boxed{T_3} & \boxed{T_4} & \boxed{12} \end{bmatrix}$, each value represents the temperature of a square tile, and should be equal to the average of the surrounding tile temperatures. What are the values T_1 to T_4?)
- **8 9**
  9 10

1. $M = [1\ -1/3\ -1/3\ 0;\ -1/2\ 1\ 0\ -1/2];$
2. $M = [M;\ -1/2\ 0\ 1\ -1/2;\ 0\ -1/3\ -1/3\ 1];$

- A : ~~$b = [3\ 0\ 0\ 6]';$~~ (2 … 4)
- B : ~~$b = [6\ 0\ 0\ 12]';$~~

4. reshape($M \setminus b$, [2, 2])

$$\begin{bmatrix} T_1 = \frac{1}{3}(6 + T_2 + T_3) \\ T_2 = \frac{1}{2}(T_1 + T_4) \\ T_3 = \frac{1}{2}(T_1 + T_4) \\ T_4 = \frac{1}{3}(T_2 + T_3 + 12) \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{2} & 1 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{3} & -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 4 \end{bmatrix}$$

Linear System, Temperature, Homework

Quiz

Invertability

Math

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$M^{-1} M = I$$

- A square matrix M is invertible if there exists M^{-1} such that $M^{-1}M = I$, and M is $\boxed{\text{singular}}$ if it is not invertible.
- If M is invertible, then the solution to $Mx = b$ is $x = M^{-1}b$.
- If M is singular, then $Mx = b$ may not have a solution for x.

$$\underbrace{\cancel{M^{-1}M}}_{I} x = M^{-1} b$$

$$? \; M = 1$$

$$? = \frac{1}{M} = M^{-1}$$

Determinant

Math

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $ad - bc = 0$

- The determinant of a matrix M, denoted by $\det(M)$ or $|M|$, measures the magnitude of a matrix.
- $\det(M) = 0$ if and only if M is singular.
- When $\det(M)$ is close to 0, M could be difficult to invert due to numerical errors, and MATLAB issues a warning about the matrix being close to singular.

$\begin{bmatrix} 0 \end{bmatrix}$

$\frac{1}{\text{small}}$

$[+5 \cancel{\approx}]$ numerical

Condition Number

Math

- Matrices that are close to singular are also called ill-conditioned.
- The condition number of a matrix M, denoted by $\kappa(M)$, measures how much the solution x changes due to a small error in b.
- The larger the condition number, the more sensitive the solution is to the changes in b, which implies that numerical errors are more likely to affect the solution.
- If M is not invertible, the condition number of M is infinity.

$Mx = b$

Inversion and Condition Number

Code

- *inv(M)* finds the inverse of a square matrix M.
- *det(M)* finds the determinant of a square matrix M.
- *cond(M)* finds the condition number of a matrix M.

Hilbert Matrix

Math

- Hilbert matrix is an example of an ill-conditioned matrix.
- Row i column j of a Hilbert matrix is $H_{ij} = \dfrac{1}{i+j-1}$.
- For example, a 3 by 3 Hilbert matrix is $\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$,

  and a 4 by 4 Hilbert matrix is $\begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{bmatrix}$.
- *hilb (n)* creates the n by n Hilbert matrix.

Condition Number, Hilbert Matrix

Quiz

Q6

$$\underset{m}{\begin{bmatrix} 1 & \frac{1}{2} & \\ & \ddots & \\ & & \end{bmatrix}} \cdot \underset{1}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix}} = \underset{b}{\begin{bmatrix} b \end{bmatrix}}$$

1. *m = hilb(5); b = m ∗ ones(5, 1); x = m \ b;*
2. *cond(m)* %4.7661e+05
3. *[min(x) max(x)]*
   - *B* : **1 1**
   - *C* : **0 0**

$$b / M = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

1 1

Condition Number, Hilbert Matrix Large

Quiz

$$\begin{bmatrix} m \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = b$$

1. $m = hilb(25);\ b = m * ones(25, 1);\ x = m \setminus b;$
2. $cond(m)$ %8.9640e+18
3. $[min(x)\ max(x)]$ $\neq$

- B : **1 1**
- C : **-104.7468 74.0750**

$x \neq 1$

b/m

Condition Number, Hilbert Matrix Larger

Quiz

always check cond
if large
do not
trust
the solution
from MATLAB

1. $m = hilb(100); \; b = m * ones(100, 1); \; x = m \setminus b;$
2. $cond(m)$ %5.6675e+19
3. $[min(x) \; max(x)]$

- B : **1 1**
- C : **-328.5181 400.4187**

$m \setminus b =$

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