Guess Two-Thirds of the Average Game Quiz

- Enter an integer between 0 and 100 (including 0 and 100) that is the closest to \( \frac{2}{3} \) of the average of everyone’s integer.
A Simple Linear System

Quiz

What is the height of the table?
System of Two Equations
Math

\[
\begin{align*}
    m_{11}x_1 + m_{12}x_2 &= b_1 \\
    m_{21}x_1 + m_{22}x_2 &= b_2
\end{align*}
\]
is a system of two equations and two unknowns \(x_1\) and \(x_2\).

The system can be written in matrix form
\[
\begin{bmatrix}
    m_{11} & m_{12} \\
    m_{21} & m_{22}
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix}
= 
\begin{bmatrix}
    b_1 \\
    b_2
\end{bmatrix}
\]

The system may have 0, 1 or infinite number of solutions.
Two Equations Example

Math

- \([1 2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}] = \begin{bmatrix} 4 \\ 5 \end{bmatrix}\) has no solution.

- \([1 2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}] = \begin{bmatrix} 4 \\ 12 \end{bmatrix}\) has infinite number of solutions,

  \[x = \begin{bmatrix} 4 - 2t \\ t \end{bmatrix}, \quad t \in \mathbb{R}\.

- \([1 2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}] = \begin{bmatrix} 5 \\ 11 \end{bmatrix}\) has a unique solution \(x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\).
In general, for
\[
\begin{bmatrix}
  m_{11} & m_{12} \\
  m_{21} & m_{22}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}
= \begin{bmatrix}
  b_1 \\
  b_2
\end{bmatrix},
\]
if
\[d = m_{11}m_{22} - m_{12}m_{21} = 0\]
then there are either no solution or infinite number of solutions; otherwise, there is a unique solution
\[
x = \frac{1}{d} \begin{bmatrix}
  m_{22}b_1 - m_{12}b_2 \\
  m_{11}b_2 - m_{21}b_1
\end{bmatrix}.
\]
Solving Two Equations

Code

- \([m_{11} \ m_{12}; \ m_{21} \ m_{22}] \ \backslash \ [b_1; \ b_2] \text{ or} \ [b_1; \ b_2] / [m_{11} \ m_{12}; \ m_{21} \ m_{22}]\) solves the system
  \[
  \begin{bmatrix}
  m_{11} & m_{12} \\
  m_{21} & m_{22}
  \end{bmatrix}
  \begin{bmatrix}
  x_1 \\
  x_2
  \end{bmatrix}
  =
  \begin{bmatrix}
  b_1 \\
  b_2
  \end{bmatrix}
  \text{ for } \begin{bmatrix}
  x_1 \\
  x_2
  \end{bmatrix}.
  \]

- The MATLAB command will always output a solution, usually close to infinity if there are none; and only output one if there are infinite number of them.

- There will be a warning about the matrix being singular.
System of Equations

Math

\[
\begin{align*}
    m_{11}x_1 + m_{12}x_2 + \ldots + m_{1n}x_n &= b_1 \\
    m_{21}x_1 + m_{22}x_2 + \ldots + m_{2n}x_n &= b_2 \\
    \vdots \\
    m_{k1}x_1 + m_{k2}x_2 + \ldots + m_{kn}x_n &= b_k
\end{align*}
\]

is a system of \( k \) equations and \( n \) unknowns \( x_1, x_2, \ldots, x_n \).

The system can be written in matrix form

\[
\begin{bmatrix}
    m_{11} & m_{12} & \ldots & m_{1n} \\
    m_{21} & m_{22} & \ldots & m_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    m_{k1} & m_{k2} & \ldots & m_{kn}
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_n
\end{bmatrix}
= 
\begin{bmatrix}
    b_1 \\
    b_2 \\
    \vdots \\
    b_k
\end{bmatrix}, \text{ or } Mx = b.
\]
Solving a System

Math

- Let $Mx = b$ be a system of linear equations.
- $x$ can be solved by hand using Gaussian elimination.
- It is faster to solve for $x$ using LU decomposition.
LU Decomposition

Math

- Every matrix $M$ can be written in the form $PM = LU$.

  1. $P$ is a permutation matrix (each row and each column contain one 1 and all other entries are 0s).

  2. $L$ is a lower triangular matrix (all entries above the diagonal are 0s).

  3. $U$ is a upper triangular matrix (all entries below the diagonal are 0s).

- $Mx = b$ can be solved by solving $Ly = Pb$ for $y$ then solving $Ux = y$ for $x$. 
Two by Two LU Decomposition

Math

- $M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ can be written as

  \[ LU = \begin{bmatrix} 1 & 0 \\ m_{21} & 1 \\ m_{11} & \frac{1}{m_{11}} \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ 0 & m_{11}m_{22} - m_{12}m_{21} \end{bmatrix} \]

- Here, $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the permutation matrix assuming $m_{11} \neq 0$, and if $m_{11} = 0$, $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ can be used instead.

- This makes solving $Ly = b$ and $Ux = y$ very fast since only forward and backward substitutions are required.
Solving
Code

- \( M \backslash b \) or \( b \div M \) solves \( Mx = b \) for \( x \).
- \([L, U, P] = \text{lu}(M)\) computes the LU decomposition and \( y = L \backslash (P \ast b) \); \( x = U \backslash y \) also solves \( Mx = b \) for \( x \).
- In the case \( Mx = b \) needs to be solved repeatedly for the same \( M \) but different \( b \)'s, it is faster to find the LU decomposition once and use \( L, U, P \) on different \( b \)'s.
- \( d = \text{decomposition}(M, 'lu'); \ d \backslash b \) uses the same LU decomposition approach without the need to remember how to solve for \( x \) given the decomposition.
Linear System, Simple
Quiz

(Solve $x_1 - x_2 = 0$ and $x_1 + x_2 = 2$.)

- $1 \quad 1$
- $A : \begin{bmatrix} 1 & -1; & 1 & 1 \end{bmatrix} \ \begin{bmatrix} 0; & 2 \end{bmatrix}$
- $B : \begin{bmatrix} 1 & 1; & -1 & 1 \end{bmatrix} \ \begin{bmatrix} 0; & 2 \end{bmatrix}$
(In a factor input matrix $M$, row $i$ column $j$ represents the amount of material $i$ required in the production of product $j$. In an input vector $b$, row $i$ represents the amount material $i$ available. Given $M$, $b$, compute the number of products that can be produced.)

- $1 \ 1 \ 1$

- $M = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{bmatrix}$; $b = \begin{bmatrix} 12 \\ 15 \\ 19 \end{bmatrix}$;

- $A : M \backslash b$

- $B : M' \backslash b$
Linear System, Temperature

Quiz

\[
\begin{bmatrix}
6 & T_1 & T_2 & - \\
- & T_3 & T_4 & 12
\end{bmatrix}, \text{ each value represents the temperature of a square tile, and should be equal to the average of the surrounding tile temperatures. What are the values } T_1 \text{ to } T_4?\]

8 9
9 10

1 \( M = \begin{bmatrix} 1 & -1/3 & -1/3 & 0; & -1/2 & 1 & 0 & -1/2 \end{bmatrix}; \)
2 \( M = [M; -1/2 0 1 -1/2; 0 -1/3 -1/3 1]; \)

\( A : b = [2 0 0 4]'; \)
\( B : b = [6 0 0 12]'; \)

4 \( \text{reshape}(M \setminus b, [2, 2]) \)
Linear System, Temperature, Homework

Quiz
Invertability

Math

A square matrix $M$ is invertible if there exists $M^{-1}$ such that $M^{-1}M = I$, and $M$ is singular if it is not invertible.

If $M$ is invertible, then the solution to $Mx = b$ is $x = M^{-1}b$.

If $M$ is singular, then $Mx = b$ may not have a solution for $x$. 
The determinant of a matrix $M$, denoted by $\det(M)$ or $|M|$, measures the magnitude of a matrix.

$\det(M) = 0$ if and only if $M$ is singular.

When $\det(M)$ is close to 0, $M$ could be difficult to invert due to numerical errors, and MATLAB issues a warning about the matrix being close to singular.
Matrices that are close to singular are also called ill-conditioned.

The condition number of a matrix $M$, denoted by $\kappa(M)$, measures how much the solution $x$ changes due to a small error in $b$.

The larger the condition number, the more sensitive the solution is to the changes in $b$, which implies that numerical errors are more likely to affect the solution.

If $M$ is not invertible, the condition number of $M$ is infinity.
Inversion and Condition Number

Code

- $\text{inv}(M)$ finds the inverse of a square matrix $M$.
- $\text{det}(M)$ finds the determinant of a square matrix $M$.
- $\text{cond}(M)$ finds the condition number of a matrix $M$. 
Hilbert Matrix

Math

- Hilbert matrix is an example of an ill-conditioned matrix.
- Row $i$ column $j$ of a Hilbert matrix is $H_{ij} = \frac{1}{i + j - 1}$.
- For example, a 3 by 3 Hilbert matrix is

$$
\begin{bmatrix}
1 & 1/2 & 1/3 \\
1/2 & 1/3 & 1/4 \\
1/3 & 1/4 & 1/5 \\
\end{bmatrix},
$$

and a 4 by 4 Hilbert matrix is

$$
\begin{bmatrix}
1 & 1/2 & 1/3 & 1/4 \\
1/2 & 1/3 & 1/4 & 1/5 \\
1/3 & 1/4 & 1/5 & 1/6 \\
1/4 & 1/5 & 1/6 & 1/7 \\
\end{bmatrix}.
$$

- `hilb(n)` creates the $n$ by $n$ Hilbert matrix.
Condition Number, Hilbert Matrix

Quiz

1. \( m = \text{hilb}(5); \quad b = m \times \text{ones}(5, 1); \quad x = m \backslash b; \)
2. \( \text{cond}(m) \ %4.7661e+05 \)
3. \( [\text{min}(x) \ \text{max}(x)] \)
   - \( B : 1 \quad 1 \)
   - \( C : 0 \quad 0 \)
Quiz

1. \( m = \text{hilb}(25); \ b = m * \text{ones}(25, 1); \ x = m \backslash b; \)
2. \( \text{cond}(m) \approx 8.9640e+18 \)
3. \([\text{min}(x) \ \text{max}(x)]\)
   - \( B : 1 \quad 1 \)
   - \( C : -104.7468 \quad 74.0750 \)
Condition Number, Hilbert Matrix Larger

Quiz

1. \( m = \text{hilb}(100); \ b = m \ast \text{ones}(100, 1); \ x = m \backslash b; \)
2. \( \text{cond}(m) \ %5.6675e+19 \)
3. \([\text{min}(x) \ \text{max}(x)]\)

- \( B : 1 \ 1 \)
- \( C : -328.5181 \ 400.4187 \)
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