CS368 MATLAB Programming

Lecture 6

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Based on lecture slides by Michael O'Neill and Beck Hasti

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System of Two Equations

Math

- $\begin{cases} m_{11}x_1 + m_{12}x_2 = b_1 \\ m_{21}x_1 + m_{22}x_2 = b_2 \end{cases}$ is a system of two equations and two unknowns x_1 and x_2.
- The system can be written in matrix form
$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$
- The system may have $0, 1$ or infinite number of solutions.

Two Equations Example

Math

- $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ has no solution.
- $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \end{bmatrix}$ has infinite number of solutions, $x = \begin{bmatrix} 4 - 2t \\ t \end{bmatrix}, t \in \mathbb{R}$.
- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$ has a unique solution $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Two Equations General Solution

Math

- In general, for $\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, if $d = m_{11}m_{22} - m_{12}m_{21} = 0$, then there are either no solution or infinite number of solutions; otherwise, there is a unique solution $x = \frac{1}{d}\begin{bmatrix} m_{22}b_1 - m_{12}b_2 \\ m_{11}b_2 - m_{21}b_1 \end{bmatrix}$.

Solving Two Equations

Code

- *[m11 m12; m21 m22] \ [b1; b2]* or *[b1; b2] / [m11 m12; m21 m22]* solves the system $\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ for $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.
- The MATLAB command will always output a solution, usually close to infinity if there are none; and only output one if there are infinite number of them.
- There will be a warning about the matrix being singular.

System of Equations

Math

- $$\begin{cases} m_{11}x_1 + m_{12}x_2 + \ldots + m_{1n}x_n = b_1 \\ m_{21}x_1 + m_{22}x_2 + \ldots + m_{2n}x_n = b_2 \\ \ldots \\ m_{k1}x_1 + m_{k2}x_2 + \ldots + m_{kn}x_n = b_k \end{cases}$$ is a system of k equations and n unknowns $x_1, x_2, \ldots, x_n$.
- The system can be written in matrix form $$\begin{bmatrix} m_{11} & m_{12} & \ldots & m_{1n} \\ m_{21} & m_{22} & \ldots & m_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ m_{k1} & m_{k2} & \ldots & m_{kn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_k \end{bmatrix}$$, or $Mx = b$.

Solving a System

Math

- Let $Mx = b$ be a system of linear equations.
- x can be solved by hand using Gaussian elimination.
- It is faster to solve for x using LU decomposition.

LU Decomposition

Math

- Every matrix M can be written in the form $PM = LU$.

1. P is a permutation matrix (each row and each column contain one 1 and all other entries are 0s).
2. L is a lower triangular matrix (all entries above the diagonal are 0s).
3. U is a upper triangular matrix (all entries below the diagonal are 0s).

- $Mx = b$ can be solved by solving $Ly = Pb$ for y then solving $Ux = y$ for x.

Two by Two LU Decomposition

Math

- $M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ can be written as
  $$LU = \begin{bmatrix} 1 & 0 \\ \dfrac{m_{21}}{m_{11}} & \dfrac{1}{m_{11}} \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ 0 & m_{11}m_{22} - m_{12}m_{21} \end{bmatrix}.$$
- Here, $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the permutation matrix assuming $m_{11} \neq 0$, and if $m_{11} = 0$, $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ can be used instead.
- This makes solving $Ly = b$ and $Ux = y$ very fast since only forward and backward substitutions are required.

Solving

Code

- $M \setminus b$ or $b\ /\ M$ solves $Mx = b$ for x.
- $[L,\ U,\ P] = lu(M)$ computes the LU decomposition and $y = L \setminus (P * b);\ x = U \setminus y$ also solves $Mx = b$ for x.
- In the case $Mx = b$ needs to be solved repeatedly for the same M but different b's, it is faster to find the LU decomposition once and use L, U, P on different b's.
- $d = decomposition(M,\ 'lu');\ d \setminus b$ uses the same LU decomposition approach without the need to remember how to solve for x given the decomposition.

Linear System Quiz Questions

Quiz

Invertability

Math

- A square matrix M is invertible if there exists M^{-1} such that $M^{-1}M = I$, and M is singular if it is not invertible.
- If M is invertible, then the solution to $Mx = b$ is $x = M^{-1}b$.
- If M is singular, then $Mx = b$ may not have a solution for x.

Determinant

Math

- The determinant of a matrix M, denoted by $\det(M)$ or $|M|$, measures the magnitude of a matrix.
- $\det(M) = 0$ if and only if M is singular.
- When $\det(M)$ is close to 0, M could be difficult to invert due to numerical errors, and MATLAB issues a warning about the matrix being close to singular.

Condition Number

Math

- Matrices that are close to singular are also called ill-conditioned.
- The condition number of a matrix M, denoted by $\kappa(M)$, measures how much the solution x changes due to a small error in b.
- The larger the condition number, the more sensitive the solution is to the changes in b, which implies that numerical errors are more likely to affect the solution.
- If M is not invertible, the condition number of M is infinity.

Inversion and Condition Number

Code

- *inv(M)* finds the inverse of a square matrix M.
- *det(M)* finds the determinant of a square matrix M.
- *cond(M)* finds the condition number of a matrix M.

Hilbert Matrix

Math

- Hilbert matrix is an example of an ill-conditioned matrix.
- Row i column j of a Hilbert matrix is $H_{ij} = \dfrac{1}{i+j-1}$.
- For example, a 3 by 3 Hilbert matrix is $\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$,

  and a 4 by 4 Hilbert matrix is $\begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{bmatrix}$.
- $hilb\,(n)$ creates the n by n Hilbert matrix.

Condition Number Quiz Questions

Quiz

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